1) The events A, B, and C here are independent. Suppose that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, and $P(C) = \frac{1}{2}$ (10 points).

a) Find $P(A \cup B \cup C)$

$$\begin{aligned}
P(A \cup B \cup C) &= P(A) + P(B) + P(C) \\
&\quad - P(A \cap B) - P(A \cap C) - P(B \cap C) \\
&\quad + P(A \cap B \cap C) \\
&= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \\
&\quad - \frac{1}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} \\
&\quad + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\
&= \frac{7}{8}
\end{aligned}$$

b) Calculate $P(A \cap B^c)$

$$P(A \cap B^c) = P(A)P(B^c) = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

OR

$$\begin{aligned}
P(A \cap B^c) &= P(A) - P(A \cap B) \\
&= \frac{1}{2} - \frac{1}{2} \times \frac{1}{2} \\
&= \frac{1}{4}
\end{aligned}$$

This problem can also be solved by drawing a Venn Diagram with 3 circles; every region in this diagram contibutes $\frac{1}{8}$ probability.

2) On a game show, three boxes each contain 25 envelopes. In the first box 24 of the envelopes contain \$100. The second box contains 15 envelopes with \$100 and the third box has 12 envelopes that contain \$100. The rest of the envelopes are empty. A contestant rolls a fair die and selects an envelope at random from the first box if he rolls a 1, from the second box if he rolls a 2 or 3, and from the third box if he rolls a 4, 5, or 6 (10 points).

a) Find the probability that the contestant chose an envelope from the first box, given that he won \$100.

This is a standard Bayes' Rule problem.

$$\frac{\frac{1}{6} \times \frac{24}{25}}{\frac{1}{6} \times \frac{24}{25} + \frac{2}{6} \times \frac{15}{25} + \frac{3}{6} \times \frac{12}{25}}$$

b) Find the probability that the contestant did not win \$100 given that he selected the third box.

For this, one need not use Bayes' Rule. The easiest solution involves restricting the sample space to the event that the third box has been selected. Now 13 envelopes don't contain the \$100, and so, the probability is

$$\frac{13}{25}$$

3) Six balls are selected at random without replacement from an urn containing three white balls and five blue balls. Find the probability that at least two of the balls selected are white. (5 points).

$$\frac{C(3,2) \times C(5,4) + C(3,3) \times C(5,3)}{C(8,6)}$$

There are $C(8,6)$ ways of selecting 6 balls from 8 (denominator). Either two white and 4 blue balls are selected or 3 white and 3 blue balls are selected (numerator).

4) Two balanced dice are rolled and the numbers on the uppermost faces are noted (10 points).

a) Calculate the probability that one even and one odd number appears.

$$\frac{3 \times 3 + 3 \times 3}{36} = \frac{1}{2}$$

The sample space contains 36 elements (the denominator). Either the first number is odd and the second is even or vice versa (the numerator).

b) Find the probability that the product of the numbers is odd.

The product is odd if and only if both the numbers are odd.

$$\frac{3 \times 3}{36} = \frac{1}{4}$$

5) In a class of 16 students, what is the probability that at least two of the students share the same Birthday? Assume a 365-day calendar year (5 points).

It is easier to calculate the probability of the complementary event—all the students have different birthdays.

$$1 - \frac{P(365, 16)}{(365)^{16}} \approx 0.283604005$$

6) Two cards are randomly drawn in succession without replacement from a standard deck of 52 cards (10 points).

a) What is the probability that the first card is a face card (Jack, Queen, or King) given that the second card is a King?

$$\frac{\frac{11 \times 4}{52 \times 51}}{\frac{51 \times 4}{52 \times 51}} = \frac{11}{51}$$

b) Find the probability that at least one face card is selected given that both cards belong to the same suite.

$$\frac{\frac{2 \times 12 \times 10 + 12 \times 2}{52 \times 51}}{\frac{52 \times 12}{52 \times 51}} = \frac{22}{52}$$