1) \( V_{\text{L}} + 0.8 = 4.8991 \, \text{MHz} \) [lowest transition, \( \ell = 0 \rightarrow 1 \)]

\[ \nu = 2\pi \left( \frac{l+1}{l+1} \right) = 1 \]

So, \( 4.8991 \, \text{MHz} = 2B \)

\[ B = 2.4495 \, \text{MHz} \]

\[ B = \frac{\hbar}{8\pi^2 I} \] \text{ we need to calculate } I. \text{ Then } I = \frac{m c^2}{\hbar}.

\[ \mu = \frac{m(c^2) m_{c^2}}{m_{c^2} + m_{c^2}} = \frac{12.0000 \cdot 31.397267}{12.0000 + 31.397267} = 8.725194 \, \text{amu} \]

\[ \mu = 8.725194 \, \text{amu} \times 1.66056 \times 10^{-27} \, \text{kg} \]

\[ \mu = 1.48897 \times 10^{-26} \, \text{kg} \]

\[ 2.4495 \times 10^4 \, \text{Hz} = \frac{6.6268 \times 10^{-34} \text{Js}}{8\pi^2 \left( \frac{1.48897 \times 10^{-26} \, \text{kg} \times c}{\text{amu}} \right)} \]

\[ I = 3.42599 \times 10^{-47} \, \text{kg m}^2 \]

\[ I = \mu c^2 \]

\[ \Rightarrow r^2 = 2.3645988 \times 10^{-10} \, \text{m}^2 \]

\[ r = 1.53773 \times 10^{-10} \, \text{m} \]

\( r = 1.53773 \, \text{Å} \)
\[ \nabla^2 = \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} \right) + \frac{1}{r^2} \frac{d^2}{d \theta^2} \]

\[ \nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} \frac{d^2}{d \theta^2} \]

Show these are equivalent forms.

We can drop the last term—that's the same in both cases.

\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} f \right) = \frac{1}{r} \left[ r \frac{d^2 f}{dr^2} + \frac{d f}{dr} \frac{dr}{dr} \right] = \frac{1}{r} \left[ r \frac{d^2 f}{dr^2} + \frac{df}{dr} \right] \]

\[ = \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} \Rightarrow (2) \]

Which is what we want to prove.
3) $Y_\ell^m (\theta, \phi) \Rightarrow Y_\ell^m$ so $m = l = 4$ $m = 3$

(a) Rotational energy is given by

$$E = \frac{\hbar^2}{2I} \ell (\ell + 1) \Rightarrow \frac{\hbar^2}{2I} (4)(5) = \frac{20\hbar^2}{2I} = \frac{10\hbar^2}{I}$$

(b) $L_z = m \hbar$ so $L_z = 3 \hbar$

(c) $L^2 = \hbar^2 \ell (\ell + 1)$

so $L^2 = \hbar^2 (4)(5) = 20\hbar^2$ [Note, here's a factor of 2 in front]

Be careful here, you're asked for $L$ not $L^2$

$$L = \sqrt{20} \hbar$$

These relationships are important make sure you know them.
4) $Y_l^m(\theta, \phi)$ cannot exist.
Since the spherical harmonics are $Y_l^{-m}$, this would require $l = 4$, $m = 5$.
Since $m = 0, \pm 1, \pm 2, \ldots \leq l$, $l$ cannot be less than $m$.

5) $\hat{\nabla} \omega = -i \hbar \left( \frac{\partial}{\partial \theta} - \frac{z \partial}{\partial \phi} \right)$
$\hat{\nabla} \phi = -i \hbar \left( \frac{2 \partial}{\partial x} - x \frac{\partial}{\partial \phi} \right)$

We need to compute $[\hat{\nabla}_\theta, \hat{\nabla}_\phi] f = (\hat{\nabla}_\phi \hat{\nabla}_\theta - \hat{\nabla}_\theta \hat{\nabla}_\phi) f$

(1st) Evaluate $\hat{\nabla}_\theta \hat{\nabla}_\phi f = -i \hbar \left( \frac{\partial}{\partial \theta} - \frac{z \partial}{\partial \phi} \right) \left( \frac{\partial}{\partial \phi} - \frac{x \partial}{\partial \phi} \right) f$

$= i \hbar \left( \frac{\partial}{\partial \theta} - \frac{z \partial}{\partial \phi} \right) \left( \frac{\partial}{\partial \phi} - \frac{x \partial}{\partial \phi} \right) f$

$= -\hbar^2 \left( \frac{\partial}{\partial \phi} \left( \frac{2 \partial}{\partial x} - x \frac{\partial}{\partial \phi} \right) - \frac{2 \partial}{\partial x} \left( \frac{\partial}{\partial \phi} - \frac{x \partial}{\partial \phi} \right) \right)$

$= -\hbar^2 \left( \frac{\partial}{\partial \phi} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} + \frac{x \partial^2}{\partial x \partial \phi} - \frac{x \partial^2}{\partial y \partial \phi} \right)$
\[
\frac{\hat{L}_x \hat{L}_y f}{\hbar^2} = -\hbar^2 \left( \frac{\partial^2}{\partial x^2} - \frac{x}{\partial x} \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 f}{\partial x \partial y} - \frac{x \frac{\partial f}{\partial x}}{\partial y} \right)
\]

\[
= -\hbar^2 \left( \frac{\partial^2}{\partial x^2} - \frac{x}{\partial x} \frac{\partial}{\partial x} \right) \left( \frac{\partial^2 f}{\partial y^2} - \frac{x \frac{\partial f}{\partial x}}{\partial y} \right)
\]

\[
= -\hbar^2 \left( \frac{\partial^2}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2}{\partial x \partial y} \frac{\partial^2 f}{\partial y^2} - \frac{x \frac{\partial^2 f}{\partial x \partial y}}{\partial y} - \frac{\partial^2}{\partial x \partial y} \frac{x \frac{\partial f}{\partial x}}{\partial y} \right)
\]

Now, evaluate \( \frac{\hat{L}_x \hat{L}_y f}{\hbar^2} = \hat{L}_y \hat{L}_x f \):

\[
\Rightarrow -\hbar^2 \left( \frac{\partial f}{\partial x} + \frac{x}{\partial x} \frac{\partial f}{\partial x} \right) \frac{\partial^2}{\partial x \partial y} - \frac{x \frac{\partial^2 f}{\partial x \partial y}}{\partial y} - \frac{\partial^2}{\partial x \partial y} \frac{x \frac{\partial f}{\partial x}}{\partial y} + \frac{x}{\partial x} \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2}{\partial x \partial y} \frac{x \frac{\partial f}{\partial x}}{\partial y}
\]

\[
= -\hbar^2 \left( \frac{x \frac{\partial f}{\partial x}}{\partial y} - \frac{x}{\partial x} \frac{\partial f}{\partial x} \right) = +\hbar^2 \left( x \frac{\partial f}{\partial y} - \frac{x}{\partial x} \frac{\partial f}{\partial x} \right)
\]

Which can be written as \( \hat{L} = \hat{L}_x \hat{L}_y \hat{f} \)

\[
[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z
\]