MAT 53353: ADVANCED PERSPECTIVES ON THE CALCULUS
HOMEWORK 3

Exercise 3.1. Give an example of a continuous function $f : X \to Y$ and an open set $\mathcal{O} \subseteq X$ so that $f(\mathcal{O})$ is not open. Give a second example of a continuous function and an open set such that the image of the open set under the function is closed.

Exercise 3.2. In class, we gave the following definition for a continuous function:

> Let f be defined on a (metric) topological space X. f is continuous if, for any open set $\mathcal{O} \in X$, $f^{-1}(\mathcal{O})$ is open.

In [1], Rudin gives the following definition for a continuous function:

> Let f be defined on E. Then f is said to be continuous at a point x of E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that
> $$|f(t) - f(x)| < \epsilon$$
> for all points t of E for which $|t - x| < \delta$.

Rudin goes on to say that f is continuous if it is continuous at all points where it is defined. Explain how these definitions are equvalent so long as $X = \mathbb{R}$ equipped with the metric $d(x, y) = |x - y|$.

Exercise 3.3. Give an example, different from any presented in class, of a sequence of continuous functions whose limit is not continuous.

Exercise 3.4. Demonstrate that the sequence
$$f_n(x) = \frac{n + \cos(nx)}{3n}$$
converges (pointwise) to a continuous function.

Exercise 3.5. Under what circumstances, if any, is the composition of a uniformly continuous function and a continuous function uniformly continuous?

References

[1] W. Rudin, *Principles of matheamtical analysis.*