MAT 53353: ADVANCED PERSPECTIVES ON THE CALCULUS
HOMEWORK 4

Exercise 4.1. A function $f : X \to \mathbb{R}$ is *Lipschitz continuous* if

$$|f(x) - f(y)| \leq C|x - y|$$

for some constant $C \geq 0$ and all $x, y \in X$. Show that any Lipschitz continuous function

(a) might not be differentiable, but

(b) if it is, the derivative is bounded.

Exercise 4.2. Let

$$f(t) = \begin{cases} t \sin(1/t) & t \neq 0, \\ 0 & t = 0. \end{cases}$$

$$g(t) = \begin{cases} t^2 \sin(1/t) & t \neq 0, \\ 0 & t = 0. \end{cases}$$

Use a graphing utility to examine the graphs of these functions. Then prove that one of them is differentiable at $t = 0$ while the other is not.

Exercise 4.3. What advantage does the method of differentials hold over the more traditional, limit-based approach? Conversely, what are the advantages of the limit-and-difference-quotient approach to derivatives?

Exercise 4.4. Give an example of a continuous function of x that is not differentiable at $x = 0$. Then give an example of a function of x that is differentiable *only* at $x = 0$.

Exercise 4.5. Suppose f and g are differentiable, and that $g'(x_0) \neq 0$. Show that if $f(x_0) = g(x_0) = 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}.$$