MAT 3271: Selected solutions to problem set 9

Chapter 3, Exercises:

32. The difficulty in proving the SSS criterion is that the only congruence criteria we have at this point are SAS and ASA, which both, obviously, require us to know at least one pair of congruent angles. But we are given no information about angles! What to do? Clearly, we must prove that a pair of angles is congruent by using what we know about the corresponding sides of the triangles. The only theorem we have which proves that angles are congruent given that sides are congruent is Proposition 3.10, the "base angles theorem." Therefore, our strategy has to be to set up a figure with some isosceles triangles in it. Now for the details.

    Given triangles $\triangle ABC$ and $\triangle DEF$ with $AB \simeq DE$, $BC \simeq EF$, and $AC \simeq DF$, let G be the unique point on the opposite side of $\overleftrightarrow{AC}$ from B such that $\triangle DEF \simeq \triangle AGC$ (Corollary to SAS). Since congruence is transitive for segments and angles, it immediately follows from the definition that it is transitive for triangles; therefore, it suffices to prove $\triangle ABC \simeq \triangle AGC$. Furthermore, by the definition of congruence for triangles and CA2 it follows, of course, that $AB \simeq AG$, $AC \simeq AC$, and $BC \simeq GC$. We may thus assume without loss of generality that $A = D$, $G = E$, and $C = F$. That is what the text means by "reduce to the case that $A = D$, $C = F$, and the points B and E are on opposite sides of $\overleftrightarrow{AC}$."

    By definition of opposite sides, BE intersects $\overleftrightarrow{AC}$ at a point G (not to be confused with our earlier use of a point G, now discarded). By BA3, either $G = A$, $G = C$, $A * G * C$, $G * A * C$, or $A * C * G$. The same proof works for the first two cases by exchanging the labels of A and C. Similarly, the same proof applies to the last two cases. Thus it suffices to consider three cases.

    Case 1. $A * G * C$. We have $\angle EBC \simeq \angle BEC$ and $\angle EBA \simeq \angle BEA$ by Proposition 3.10. Ray $\overrightarrow{BE}$ is between rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ by Proposition 3.7. Similarly, ray $\overrightarrow{EB}$ is between rays $\overrightarrow{EA}$ and $\overrightarrow{EC}$. So $\angle ABC \simeq AEC$ by Proposition 3.19 (Angle Addition). By SAS, $\triangle ABC \simeq \triangle AEC$.

    Case 2. $G = A$. Just apply Prop. 3.10 and SAS. (Note that it follows from the definition of congruence for triangles, the definition of a right angle, and the definition of perpendicular that $BE \perp AC$, a useful fact about the median from the apex of an isosceles triangle.)

    Case 3. Similar to Case 1, but use angle subtraction instead of angle addition.

Chapter 4, Exercises:

4. Let l and m be parallel lines, and assume line n intersects line m at point P. If n does not intersect l, then n and m are distinct lines through P that are parallel to l, contradicting Hilbert's Parallel Postulate. Thus Hilbert's Parallel Postulate implies that if a line intersects one of two parallel lines, it intersects the other. Conversely, let l be a line and P a point not on l. Suppose there are two lines through P, m and n, that are parallel to l. Then n intersects one of two parallel lines (m) but not the other (l).

5. Assume Hilbert's Parallel Postulate, and let l and m be parallel lines cut by a transversal t at points P and Q, respectively. Let R and S be points on l and m, respectively, that are on opposite sides of t, so $\angle RPQ$ and $\angle SQP$ are alternate interior angles (IA2, BA2). By CA4, there is a unique ray $\overrightarrow{PT}$ on the same side of t as R such that $\angle TPQ \simeq \angle SQP$. By the Alternate Interior

Angle Theorem, $\overleftrightarrow{PT} \parallel m$. By Hilbert's Parallel Postulate, $\overleftrightarrow{PT} = l = \overleftrightarrow{PR}$. Since R and T are on the same side of t, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are not opposite rays, so $\overrightarrow{PT} = \overrightarrow{PR}$. Thus $\angle TPQ = \angle RPQ$, and $\angle RPQ \simeq \angle SQP$, by CA5.

Conversely, assume the converse to the Alternate Interior Angles Theorem, let $l = \overleftrightarrow{QS}$ be a line, and let P a point not on l. Let $m = \overleftrightarrow{PR}$ and $n = \overleftrightarrow{PT}$ be lines parallel to l, where R and T are chosen on the opposite side of $\overleftrightarrow{PQ}$ from S. By the converse to the AIAT, both $\angle RPQ$ and $\angle TPQ$ are congruent to $\angle SQP$; hence, by CA5, they are congruent to each other. By the uniqueness part of CA4, $\overrightarrow{PT} = \overrightarrow{PR}$, so $m = n$ by IA1 (uniqueness).

8. Assume Hilbert's Parallel Postulate. Given $\triangle ABC$, let l be the unique line through B that is parallel to $\overleftrightarrow{AC}$. Let P and Q be points on l such that P is on the opposite side of $\overleftrightarrow{AB}$ from C and Q is on the opposite side of $\overleftrightarrow{BC}$ from A. By Proposition 4.8, the converse of the AIAT holds, so $\angle PBA \simeq \angle CAB$ and $\angle QBC \simeq \angle ACB$.

   Claim: $\overrightarrow{BC}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BQ}$. Proof: Segment AC contains no point on line l because lines $\overleftrightarrow{AC}$ and l are parallel; therefore, A and C are on the same side of l. Since A and Q are on opposite sides of line $\overleftrightarrow{BC}$, AQ intersects $\overleftrightarrow{BC}$ at a point R. By Proposition 2.1, A and R are on the same side of l. (Q is the unique point of intersection of $\overleftrightarrow{AQ}$ and l, and $A * R * Q$.) By BA4, C and R are on the same side of l, so R is on ray $\overrightarrow{BC}$ rather than its opposite ray. Thus $\overrightarrow{BC}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BQ}$ by Proposition 3.7.

   After writing the above, I realized that there was a more efficient route to this point. This often happens when working on a proof, so rather than just replace what I wrote above, I leave it for you to compare. The key to a simpler proof is the selection of initial properties used to choose points P and Q. Let P and Q be points on l such that P and A are on the same side of $\overleftrightarrow{BC}$, and Q and C are on the same side of $\overleftrightarrow{BA}$. Now it follows directly from the definition that C is in the interior of $\angle ABQ$. (Just use our assumption about Q and the fact that $\overleftrightarrow{AC} \parallel l$.) Therefore, $\overrightarrow{BC}$ is between $\overrightarrow{BA}$ and $\overrightarrow{BQ}$, and furthermore Q and A are on opposite sides of $\overleftrightarrow{BC}$ by the Crossbar Theorem. Similarly, P is on the opposite side of $\overleftrightarrow{AB}$ from C. Once again, by the converse of the AIAT, $\angle PBA \simeq \angle CAB$ and $\angle QBC \simeq \angle ACB$.

   Claim: $\overrightarrow{BP}$ and $\overrightarrow{BQ}$ are opposite rays; hence, angles $\angle PBA$ and $\angle QBA$ are supplementary. Proof: Just use the corollary to BA4 to show that P and Q are on opposite sides of line $\overleftrightarrow{BC}$. The claim follows from the definition of opposite sides and Proposition 2.1.

   $(\angle PBA)^\circ + (\angle QBA)^\circ = 180^\circ$ by Theorem 4.3 (part A(5)). $(\angle QBC)^\circ + (\angle CBA)^\circ = (\angle QBA)^\circ$ by Proposition 4.3(part A(3)). By substitution (we're just doing algebra with numbers now), $(\angle A)^\circ + (\angle B)^\circ + (\angle C)^\circ = 180^\circ$.

13. (a) Suppose segment AB has two midpoints, M and N. By Propositions 3.5 ($AB = AM \cup MB$) and 3.3, we may assume without loss of generality (that is, switching the labels of M and N if necessary) that $A * M * N * B$. By definition, $AM < AN$. By definition of midpoint, $AN \simeq NB$, so by Proposition 3.13, $AM < NB$. Similarly, $BN < MA$, contradicting Proposition 3.13.

17. (a) Hint: $\triangle AOB$ is isosceles, by definition of a circle.

(b) Hint: Let M be the midpoint of AB. It suffices to prove that $MO \perp AB$. Use congruent triangles.

18. Theorem of Thales in Euclidean geometry: An angle inscribed in a semicircle is a right angle.

    Proof: In the figure in the text, in which the center of the circle is O and points A, B, and C lie on the circle, with $B * O * C$, observe that $OB \simeq OA \simeq OC$ (definition of circle), so $\angle OBA \simeq \angle OAB$ and $\angle OCA \simeq \angle OAC$ by Proposition 3.10. Angles $\angle BOA$ and $\angle COA$ are supplementary, so their sum is $180°$ (Theorem 4.3 (part A(5)). For convenience in doing calculations, set $(\angle OBA)^\circ = (\angle OAB)^\circ = x^\circ$ and $(\angle OCA)^\circ = (\angle OAC)^\circ = y^\circ$; here, Theorem 4.3 (part A(2)) has been applied to deduce that the measures of congruent angles are equal. The sum of the angles in each of $\triangle OAB$ and $\triangle OAC$ is $180°$ (Proposition 4.11); hence, removing the angles at O we obtain $2x + 2y = 180$, from which it follows algebraically that $(\angle BAC)^\circ = (x + y)^\circ = 90^\circ$. By Theorem 4.3(part A(1)), $\angle BAC$ is a right angle.

    Theorem of neutral geometry: *If* an angle inscribed in a semicircle is a right angle, then there exists a right triangle with zero defect.

    Recall that the *defect* of a triangle is the amount by which its angle sum falls short of $180°$ (so in Euclidean geometry the defect of every triangle is zero). Note: to deduce the conclusion of this theorem, we only need to assume that there is *one* angle inscribed in a semicircle that is a right angle. It is not necessary to assume that every angle inscribed in a semicircle is a right angle (although later we will be able to prove that is the case, if one is).

    Proof: Let $\angle BAC$ be a right angle inscribed in a semicircle. We claim that the defect of $\triangle BAC$ is zero; that is, its angle sum is $180°$. This time around, we *cannot assume* the angle sum of any triangle is $180°$, but labeling the angle measures as before, we see from the given and Theorem 4.3 that $x + y = 90$ (since $(x + y)^\circ$ is the measure of $\angle BAC$, which is assumed to be a right angle). Thus, reversing the calculation above, we see that $2x + 2y = 180$. Since $(\angle B)^\circ = x^\circ$ and $(\angle C)^\circ = y^\circ$, the defect of $\triangle BAC$ is zero.