The Geometry of Forces Along Equidistant Particle Paths

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Abstract

Assume that two particles on the sphere leave the equator moving due south and travel at a constant and equal speed along a geodesic colliding at the south pole. An observer who is unaware of the curvature of the space will conclude that there is an attractive force acting between the particles. On the other hand, if particles travel at the same speed (initially parallel) along geodesics in the hyperbolic plane, then the particle paths diverge. Imagine two particles in the hyperbolic plane that are bound together at a constant distance with their center of mass traveling along a geodesic path at a constant velocity, then the force due to the curvature of the space acts to break the bond and increases as the velocity increases. We will give the formula for the apparent force between the particles induced on 2 dimensional space forms of non-zero curvature.

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1. Introduction

In this paper we wish to study the apparent force on particles traveling in a two dimensional space with constant sectional curvature. It is well known that geodesics in positively curved space will tend to converge. On the other hand, particles traveling along geodesic paths in negatively curved space will tend to diverge.

Even though we will restrict our attention to particles traveling on surfaces, it should be the case that these results will apply to particles traveling in higher dimensional space provided that the problem closely approximates a space of constant curvature.

Let M denote a 2-dimensional non-trivial space form (i.e. Riemannian manifold with constant non-zero sectional curvature) with Riemannian metric $g(,)$. We will assume that the mass m is constant unless stated otherwise. Let $\sigma(s)$ denote a geodesic path in the manifold M such that $\sigma(0) = x_0$ and

$$\frac{d}{ds}\sigma(s)|_{s=0} = \mathbf{u}_o$$

where the magnitude of $\mathbf{u}_o$ is 1. In fact, the speed of a geodesic path is always assumed to be 1. We define an *inertial path*, $\mu(t)$, as a path along a geodesic such that the speed is constant and equal to the initial speed v_o. In other words:

$$\frac{d}{dt}\mu(t) = v_o\frac{d}{dt}\sigma(t),$$

where $\sigma(t)$ is a geodesic. We say that a path is a constant speed path if it has no linear acceleration component, i.e.

$$\|\frac{d\lambda}{dt}\| = \alpha,$$

where α is a constant. Let $\lambda(t)$ denote a constant speed path. The external force required for the particle to follow this path is given by

$$\mathbf{F} = m\frac{d^2\lambda}{dt^2}.$$

An example of such a path is given by the path of a orbiting satellite in a flat space. The external force of gravity gives the satellite a path which is not geodesic with respect to the usual metric (i.e. the flat metric). Let p_1 and

p_2 denote particles with mass m traveling along constant speed paths $\lambda_1(t)$ and $\lambda_2(t)$ respectively, such that

$$\rho(\lambda_1(t), \lambda_2(t)) = d$$

where $\rho(,)$ denotes the minimum distance in the manifold M and d is a constant. Assume that the midpoint connecting the particles at each point travels along a geodesic path. In other words, the particles move along constant speed paths at a fixed distance from the center of mass (i.e. midpoint), while the center of mass moves along an inertial path. If the particles were connected by a rigid rod of zero mass then they would exhibit this behavior. We will say that such a pair of particles along with their respective paths are *coupled*. We choose a oriented frame $\{e_1, e_2\}$ such that e_1 is a unit vector in the direction of the velocity vector for the midpoint in a coupled system. We shall refer to the direction associated to e_2 as up and its opposite as down. The force required to keep each on its path is the *coupling force*. This is precisely the tension in some imaginary connecting rod. The coupling force is positive when paths are convergent and negative when the paths are divergent. The coupling force is defined by:

$$\mathbf{F_c}(t) = m\frac{d^2\lambda_1}{dt^2} - m\frac{d^2\lambda_2}{dt^2}.$$

We will prove the following theorems:

Theorem 1. *Let M be a space form with positive curvature $K = 1/r^2 > 0$, then the coupling force between two coupled particles of mass m moving at speed v at a distance $d/2$ from a central inertial path is outward and is given by*

$$F_c = \frac{2mv^2}{r}\tan(d/2r) = 2mv^2\sqrt{K}\tan(\sqrt{K}d/2)$$

Theorem 2. *Let M be a space form with negative curvature $K = -k^2 < 0$, then the coupling force between two particles of mass m moving at speed v at a distance $d/2$ from the central inertial path is inward and is given by*

$$F_c = -2mv^2 k\tanh(kd/2).$$

We note that this force is essentially a result of the geometry of the manifold. This appears to be rather obvious although the authors are not aware that such an observation has been made in a mathematical context. We have

considered only space forms in the development here since the mathematics is much simpler than in the general case. Theorems of this kind should be possible in spaces of strictly positive or negative sectional curvature, but in these cases, choosing paths with constant speed that are equidistant becomes problematic. It seems that some sort of linear acceleration might be necessary to keep the particles equidistant.

2. Proof of Theorem 1

In this section we consider the case $M = S^2$ with sectional curvature $K = 1/r^2$. We observe that every pair of distinct geodesics converge on S^2. As an example of this, consider two particles which leave the equator of the sphere traveling due south at a constant and equal speed. The partcles will collide at the south pole. In other words, the corresponding inertial paths collide at the intersection point of the two geodesics determined by these paths. Therefore, the paths are decreasing in distance as time goes on.

In the more general setting such that the sectional curvature of a complete manifold is positive and bounded below by $\alpha > 0$ we have that conjugate points on geodesics will occur at a distance less than $\pi/\sqrt{\alpha}$ [cf. 1, pg 74]. In addition, a Jacobi field exists perpendicular to the geodesic which vanishes at the conjugate points. The Jacobi field induces a family of geodesics defining the congugate points. It is evident from this that paths with parallel initial vectors whose inital points are sufficiently close will collide though in this case it is not clear that the particles themselves will necessarily collide. It should be possible to choose a variation of the speed to effect the collision however.

Nevertheless, as the maximum distance between the induced geodesics is decreased the case of the space of constant sectional curvature will be approximated. Then as the geodesics converge a compensating force must be applied externally to the two particles to keep them at a constant distance from each other. In other words, the particles can not follow inertial paths while maintaining a constant distance. We shall calculate this force in the case of constant sectional curvature $K = 1/r^2$.

Let
$$\mu(t) = r(\cos(\theta)\cos(\omega t)\hat{\mathbf{i}} + \cos(\theta)\sin(\omega t)\hat{\mathbf{j}} + \sin(\theta)\hat{\mathbf{k}}).$$

4

denote a path in S^2 with constant speed v, where the angular velocity (with respect to the center of the circle) is $\omega = v/(r\cos(\theta))$. Note that this path is just a circlular curve at latitude θ on the sphere in the usual sense. We begin by computing the second derivative in $\mathbb{R}^3$:

$$\frac{d^2\mu}{dt^2} = -r\omega^2\cos(\theta)\cos(t)\hat{\mathbf{i}} - r\omega^2\cos(\theta)\sin(t)\hat{\mathbf{j}}.$$

This vector must be projected onto the tangent space in the plane. The corresponding tangent vector is given by:

$$\mathbf{T}_\perp = -\sin(\theta)\cos(t)\hat{\mathbf{i}} - \sin(\theta)\sin(t)\hat{\mathbf{j}} + \cos(\theta)\hat{\mathbf{k}}.$$

We obtain the force equation:

$$\mathbf{F} = m\left(\frac{d^2\mu}{dt^2}\cdot\mathbf{T}_\perp\right)\mathbf{T}_\perp,$$

$$\mathbf{F} = m\frac{rv^2}{r^2\cos^2(\theta)}(\sin(\theta)\cos(\theta)\cos^2(t) + \sin(\theta)\cos(\theta)\sin^2(t))\mathbf{T}_\perp,$$

$$\mathbf{F} = m\frac{v^2}{r}\tan(\theta)\mathbf{T}_\perp$$

Consequently the coupling force for two particles with center of mass on a great circle is:

$$F_c = \frac{2mv^2}{r}\tan(\theta).$$

Recall that the distance along an arc is given by $d = r\Delta\theta$ and $K = 1/r^2$ so we have

$$F_c = 2mv^2\sqrt{K}\tan\left(\frac{\sqrt{K}d}{2}\right).$$

Let W denote the kinetic energy then we have

$$F_c \approx mv^2Kd = 2WKd,$$

when d is sufficiently small.

3. Proof of Theorem 2

In this section we consider the case of a space form M with sectional curvature $K = -1/k^2$.

We define the Minkowski space

$$\mathbb{R}^{2,1} = \{(x, y, z)| \quad x, y, z \in \mathbb{R}\}$$

such that the distance between points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ is given by

$$\rho(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 - (z_1 - z_2)^2}.$$

Then the hyperbolic plane $\mathcal{H}^2(-k)$ is given by the set locus

$$x^2 + y^2 - z^2 = -\frac{1}{k^2}$$

or equivalently,

$$x^2 + y^2 + \frac{1}{k^2} = z^2$$

subject to the induced metric. We obtain a coordinate system for the hyperbolic plane under the projection

$$\Pi : \mathbf{R}^{2,1} \longrightarrow \mathbf{R}^2$$

where

$$\Pi(x, y, z) \mapsto (x, y).$$

Now define the linear transformation from $\mathbf{R}^{2,1}$ onto $\mathbf{R}^{2,1}$ by the matrix

$$A = \begin{pmatrix} \sqrt{k^2\alpha^2 + 1} & 0 & \alpha k \\ 0 & 1 & 0 \\ \alpha k & 0 & \sqrt{k^2\alpha^2 + 1} \end{pmatrix},$$

where the matrix acts on column vectors of $\mathbf{R}^{2,1}$. This is an isometry on the submanifold $\mathcal{H}^2(-k)$ which leaves the space $\{(x, 0, \sqrt{x^2 + 1/k^2})| x \in \mathbb{R}\}$ invariant. Similarly, the isometry

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{k^2\beta^2 + 1} & \beta k \\ 0 & \beta k & \sqrt{k^2\beta^2 + 1} \end{pmatrix},$$

6

leaves the space $\{(0, y, \sqrt{y^2 + 1/k^2}) | y \in \mathbb{R}\}$ invariant. On the other hand, we observe that the action of A on the coordinate line associated to the y-axis yields

$$A : \begin{pmatrix} 0 \\ y \\ \frac{1}{k}\sqrt{k^2y^2 + 1} \end{pmatrix} \mapsto \begin{pmatrix} \alpha\sqrt{k^2y^2 + 1} \\ y \\ \frac{1}{k}\sqrt{(k^2\alpha^2 + 1)(k^2y^2 + 1)} \end{pmatrix}.$$

If we compose this isometruy with the coordinate projection operator then the image of the y-axis $(\alpha\sqrt{k^2y^2 + 1}, y)$ satisfies

$$\frac{x^2}{\alpha^2} - \frac{y^2}{(1/k^2)} = 1.$$

In other words, if we think of $\mathbb{R}^2$ as the local coordinate system for the hyperbolic plane then the mapping takes the geodesic associated to the y-axis into a hyperbola as shown above. The action of A is similar to a translation in the direction of the x-axis.

Now consider the composition of A with a similar translation in the y-axis direction:

$$B \circ A \begin{pmatrix} 0 \\ y \\ \sqrt{k^2y^2 + 1} \end{pmatrix} = B : \begin{pmatrix} \alpha\sqrt{k^2y^2 + 1} \\ y \\ \frac{1}{k}\sqrt{(k^2\alpha^2 + 1)(k^2y^2 + 1)} \end{pmatrix} \mapsto$$

$$\begin{pmatrix} \alpha\sqrt{k^2y^2 + 1} \\ y\sqrt{\beta^2k^2 + 1} + \beta\sqrt{(k^2\alpha^2 + 1)(k^2y^2 + 1)} \\ \beta ky + \sqrt{(k^2\beta^2 + 1)(k^2\alpha^2 + 1)(y^2 + 1/k^2)} \end{pmatrix},$$

In particular when $y = 0$ we obtain

$$B : \begin{pmatrix} \alpha \\ 0 \\ \frac{1}{k}\sqrt{(k^2\alpha^2 + 1)} \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ \beta\sqrt{(k^2\alpha^2 + 1)} \\ \sqrt{(k^2\beta^2 + 1)(k^2\alpha^2 + 1)} \end{pmatrix}$$

7

In other words,

$$A \circ B : \begin{pmatrix} 0 \\ 0 \\ 1/k \end{pmatrix} \mapsto \begin{pmatrix} \alpha \\ \beta\sqrt{k^2\alpha^2 + 1} \\ \sqrt{(k^2\beta^2 + 1)(k^2\beta^2 + 1)} \end{pmatrix}$$

The critical factor is this: the origin $(0,0)$ is moved by the isometry A to the point $(\alpha, 0)$. The x-axis is the geodesic connecting the point $(0,0)$ to the point $(\alpha, 0)$. The isometry B maps this geodesic segment (i.e. the segment along the x-axis) to a geodesic segment connecting $(0, \beta)$ to $(\alpha, \beta\sqrt{k^2\alpha^2 + 1})$ The minimum distances between geodesics is preserved and therefore, the distance between the y-axis and the point $(\alpha, \beta\sqrt{k^2\alpha^2 + 1})$ is the same as the distance between $(0,0)$ and $(\alpha, 0)$.

A parametric equation for this curve of constant distance from the y-axis is given by

$$\sigma(t) = \begin{pmatrix} \alpha \\ t \\ \sqrt{\alpha^2 + t^2 + 1/k^2} \end{pmatrix}.$$

Two curves of this type symmetric with the y-axis will give a pair of inertial paths that are equidistant for all t. However, it is convenient to calculations at the origin of the coordinate system. Since $\mathcal{H}(-k)$ is a space form it suffices to translate the curve $\sigma(t)$ back to the origin using the isometry A^{-1}. We shall construct a curve with speed v at the origin such that the linear acceleration is zero at the origin. The curve is given by

$$\gamma(t) = A^{-1} \begin{pmatrix} \alpha \\ vt \\ \sqrt{\alpha^2 + v^2t^2 + 1/k^2} \end{pmatrix} \mapsto$$

$$\begin{pmatrix} \alpha\sqrt{k^2\alpha^2 + 1} - \alpha\sqrt{k^2\alpha^2 + k^2v^2t^2 + 1} \\ vt \\ -\alpha^2 k + \sqrt{(k^2\alpha^2 + 1)(k^2(\alpha^2 + v^2t^2) + 1)}/k \end{pmatrix}$$

8

and

$$\frac{d\gamma(t)}{dt} = \begin{pmatrix} -\alpha k^2 v^2 t / \sqrt{k^2(\alpha^2 + v^2 t^2) + 1} \\ v \\ tv^2 k \sqrt{k^2\alpha^2 + 1} / \sqrt{k^2(\alpha^2 + v^2 t^2) + 1} \end{pmatrix}.$$

Finally let $\eta(t) = \sqrt{k^2(\alpha^2 + v^2 t^2) + 1}$, then

$$\frac{d^2\gamma(t)}{dt^2} = \frac{1}{\eta^3(t)} \begin{pmatrix} -\alpha v^2 k^2 \eta^2(t) + \alpha k^4 v^4 t^2 \\ 0 \\ kv^2(\alpha^2 k^2 + 1)^2 \end{pmatrix}.$$

Consequently, at t = 0, we obtain $\eta(0) = \sqrt{k^2\alpha^2 + 1}$ and

$$\frac{d^2\gamma(0)}{dt^2} = \begin{pmatrix} -\alpha k^2 v^2 / \sqrt{k^2\alpha^2 + 1} \\ 0 \\ kv^2 \sqrt{\alpha^2 k^2 + 1} \end{pmatrix}.$$

Therefore, along the hyperbolic plane $\mathcal{H}(-k)$ we have

$$||\frac{d^2\gamma(0)}{dt^2}|| = \alpha k^2 v^2 / \sqrt{k^2\alpha^2 + 1}$$

On the other hand, the arc-length from the point $(0,0)$ to the point $(\alpha, 0)$ can be obtained using the arc-length integral along the path

$$r(t) = (t, 0, \sqrt{t^2 + 1/k^2}).$$

Note that

$$\frac{dr(t)}{dt} = (1, 0, kt/\sqrt{k^2 t^2 + 1}),$$

such that

$$||\frac{dr(t)}{dt}||^2 = 1 - \frac{k^2 t^2}{k^2 t^2 + 1} = \frac{1}{k^2 t^2 + 1}$$

which yields the integral

$$s = \int_0^\alpha \frac{dt}{\sqrt{1 + k^2 t^2}}$$

9

with solution
$$s = \frac{1}{k} \ln(\sqrt{k^2\alpha^2 + 1} + k\alpha)$$
or
$$\alpha = \frac{1}{k} \sinh(sk).$$

This leads to the following result,
$$\left\| \frac{d^2\gamma(0)}{dt^2} \right\| = \frac{\sinh(sk)}{\sqrt{\sinh^2(sk) + 1}} = kv^2 \frac{\sinh(ks)}{\cosh(ks)}.$$

Therefore, if we let $s = d/2$, and take into acount the mass, then the coupling force is
$$F_c = -\frac{2mv^2 k \sinh(kd/2)}{\cosh(kd/2)} = -2mv^2 k \tanh(kd/2).$$

As before, we have
$$F_c \approx -2Wk^2 d = 2WKd,$$

where W is the kinetic energy and where d is sufficiently small.

4. Conclusion

We note that the coupling force in each case is related to the kinetic energy and is nearly a linear function of the distance for small distances. From a relativistic point of view, we can see that while the velocity is bounded, the kinetic energy is not. Thus as the kinetic energy increases so does the coupling force. In the case of a negatively curved manifold, the coupling force necessary to keep the particles together must be applied inward, but if the kinectic energy is large enough and the curvature is, say increasing, then the typical chemical binding forces between particles may be broken.

The case of positive curvature is more problematic. While the curvature of the manifold tends to push the particles together, it is clear that in three dimensions this force will be somewhat unstable and probably cause the coupled system to spin. In any case, large forces acting on coupled particles in curved spaces may cause significant instability in the usual chemical bonds.

References

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