Algebraic Properties of the Real Number System

1. Properties of Operations

(0) Addition (+) and multiplication (·, often omitted) are well-defined binary operations; that is, these operations give a unique answer that depends only on the two numbers being added or multiplied.

(1) These operations are commutative and associative. That is, for all real numbers \(a\), \(b\), and \(c\),
\[
a + b = b + a, \quad a \cdot b = b \cdot a, \quad (a+b)+c = a+(b+c) \quad \text{[hence parentheses may be omitted in this situation]}, \quad \text{and} \quad (ab)c = a(bc).
\]

(2) The distributive law applies to multiplication with respect to addition: for all real numbers \(a\), \(b\), and \(c\),
\[
a (b + c) = ab + ac.
\]

(3) There is an additive identity, denoted by 0. The additive identity is defined by the property that, for all real numbers \(x\),
\[
0 + x = x.
\]

Remark. There is only one additive identity. To see this, suppose \(0'\) is another one. Then \(0 = 0 + 0'\) (by the defining property) = \(0'\) (again by the defining property), so these two numbers are really equal.

(4) There is a multiplicative identity, denoted by 1 and defined by the property that, for all real numbers \(x\),
\[
1 \cdot x = x.
\]

Remark. There is only one multiplicative identity. The proof that this is so is analogous to the one given above for the additive identity. I suggest you provide the details.

(5) Every real number \(x\) has an additive inverse, denoted by \(-x\), defined by the property that
\[
x + (-x) = 0.
\]

We define subtraction by setting \(x - y = x + (-y)\). (Note that the operation of subtraction is just a notational convenience. We don’t really need it.)

Remark. The additive inverse of a number is unique. To show this, suppose \((-x)'\) and \((-x)''\) are both additive inverses for \(x\). We will show they are the same. Indeed, \(-x = -x + (x + (-x)')\) (by the defining properties of the additive inverse and the additive identity) = \((-x + x) + (-x)'\) (by the associative property of addition) = \((-x)\).

Remark. Although 0 is defined in terms of addition, it also has a familiar property with respect to multiplication, namely that for any real number \(x\), \(0 \cdot x = 0\). We can prove this using the existence of additive inverses, along with the distributive law. \(0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x\). To summarize: \(0 \cdot x = 0 \cdot x + 0 \cdot x\). Whatever \(0 \cdot x\) is, it has an additive inverse. Adding this inverse to both sides we obtain \(0 = 0 \cdot x\).

Remark. We can also see how additive inverses multiply; we claim that \((-x)y = -xy\) and \((-x)(-y) = xy\). (Note: although it is true that the product of a negative number and a positive number is negative and the product of two negative numbers is positive, that is not what we are proving here. We don’t know if \(x\) and \(y\) are positive or negative.) In the first instance, we have \((-x)y = (-x)y + xy + (-xy) = (-x + x)y + -xy = 0 \cdot y + (-xy) = 0 + (-xy) = -xy\). I leave it to you to justify each step and supply the proof of the second statement, which is similar.

(6) Every real number \(x\) except 0 has a multiplicative inverse, denoted by \(\frac{1}{x}\) or \(x^{-1}\) and defined by the property that \(x \cdot \frac{1}{x} = 1\). We define division by setting \(x/y = x \cdot \frac{1}{y}\). (As with subtraction, division is just a notational convenience. We don’t really need it.)
Remark. The multiplicative inverse of each number is unique, by similar reasoning to that given above. Again, I suggest you provide the details of the argument.

Remark. Note that 0 is its own additive inverse, but every other number’s additive inverse is distinct from itself. Let \( x \) be a real number that is its own additive inverse; that is, \( x + x = 0 \). Then \( x = -x \). Now \( x + x = (1 + 1)x = 2x \) (by definition of 2). Hence we have \( 2x = 0 \). Multiplying by \( \frac{1}{2} \) on both sides, and using the multiplicative property of 0 proved above, we get \( x = 0 \).

Remark. Why doesn’t 0 have a multiplicative inverse? Well, we showed above that 0 · \( x \) = 0 for any real number \( x \). \( 0 \neq 1 \), so it is not possible to have a number \( x \) such that 0 · \( x \) = 1.

Remark. We can now also show, using multiplicative inverses, that if \( x y = 0 \), then \( x = 0 \) or \( y = 0 \). This fact is used to solve quadratic equations, among other things. Assume \( x y = 0 \). We will show that if \( x \neq 0 \), then \( y = 0 \). (This is logically equivalent to showing that at least one must be 0. Of course, it is possible that both \( x \) and \( y \) are 0.) Suppose \( x \neq 0 \). Then it has a multiplicative inverse, and we have \( y = 1y = (\frac{1}{x} \cdot x)y = \frac{1}{x} \cdot (xy) = \frac{1}{x} \cdot 0 = 0 \).

2. Properties of Order

There is an order relation among real numbers by which any two may be compared. That is, for all real numbers \( x \) and \( y \), one - and only one - of the following holds: \( x < y \), \( x = y \), or \( y < x \). This is often called the trichotomy property. The order relation is also transitive: if \( x < y \) and \( y < z \), then \( x < z \). To these properties of order we now need only add two properties about how order behaves with respect to the operations of addition and multiplication. We define a number to be positive if it is greater than 0.

(7) For all real numbers \( x \), \( y \), and \( z \), if \( x < y \), then \( x + z < y + z \).

Remark. From this property, it follows that if \( x \) is positive, its additive inverse, \( -x \), must be negative, and if \( x \) is negative, its additive inverse, \( -x \), must be positive. [Hint: In the first case, add \( -x \) to both sides of the inequality \( x > 0 \); in the second case, add \( -x \) to both sides of the inequality \( x < 0 \).]

(8) For all real numbers \( x \) and \( y \), and for all positive real numbers \( z \), if \( x < y \), then \( xz < yz \).

Remark. We can prove the important property that, if \( x < y \) and \( z < 0 \), then \( xz > yz \). \( x < y \iff x - y < 0 \). Since \( z < 0 \), \( -z > 0 \). Hence \((-z)(x - y) < 0 \). But \((-z)(x - y) = -zx + (-z)(-y) = zy - zx \). Thus \( zy - zx < 0 \), from which it follows that \( zy < zx \).

Remark. Based on the properties just proven, you should easily be able to show that the product of two positive numbers is positive, the product of a positive number with a negative number is negative, and the product of two negative numbers is positive. In particular, every perfect square is positive. Even more particularly, 1 must be positive, since \( 1 = 1 \cdot 1 \), and so is a perfect square. (Of course we knew that already, based on how we set up the order, but the last argument shows we could not have set things up differently, with \( 1 < 0 \), even if we had wanted to. Also, it shows that the complex numbers cannot be ordered, since \( i^2 = -1 \).)