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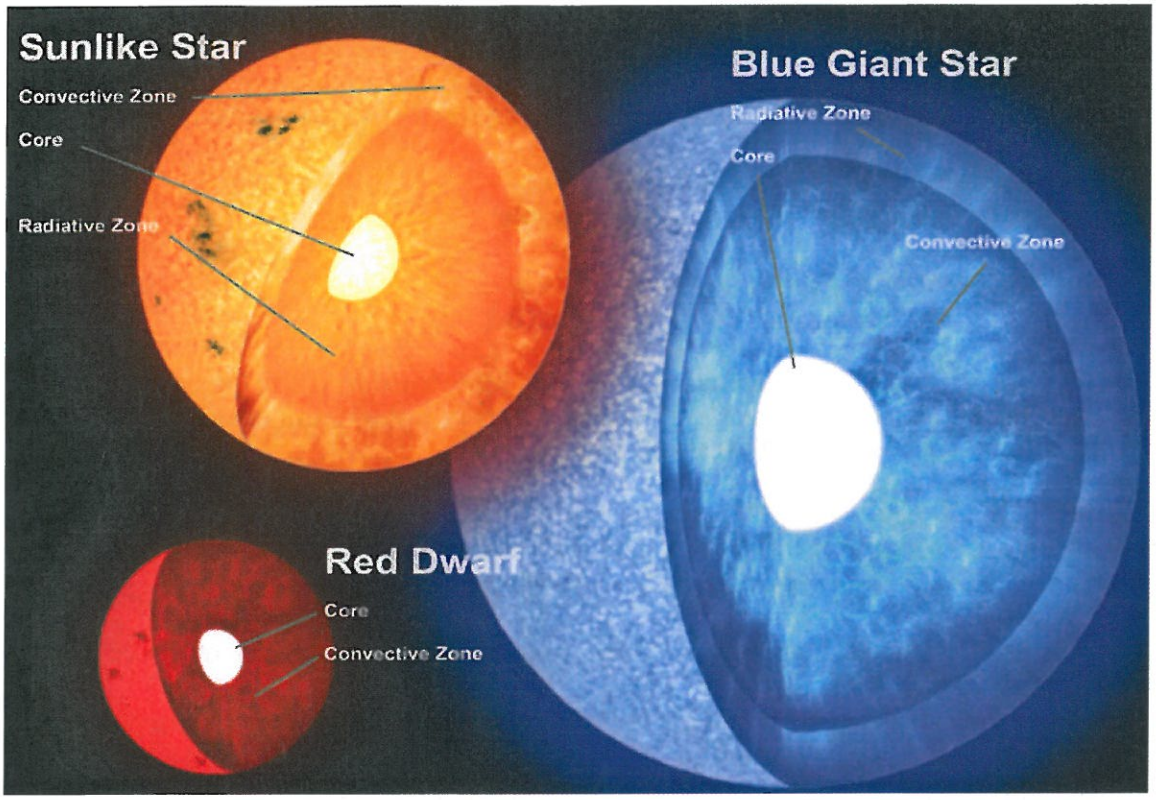
Ch. 10: Interiors of Stars



sunlike star = main sequence = burning H in core.
red dwarf = low-mass star
blue giant = high-mass star in late stage of evolution,

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Ch. 10: Interiors of Stars

- §10.1 Hydrostatic Equilibrium
- §10.2 Pressure Equation of State
- §10.3 Stellar Energy Sources
- §10.4 Energy Transport & Thermodynamics
- §10.5 Stellar Model Building
- §10.6 The Main Sequence

- Goals
  - Understand physical processes in static stars
  - Develop stellar structure equations
  - Understand how static stars are numerically modeled

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§10.1 Hydrostatic Equilibrium (p. 284)

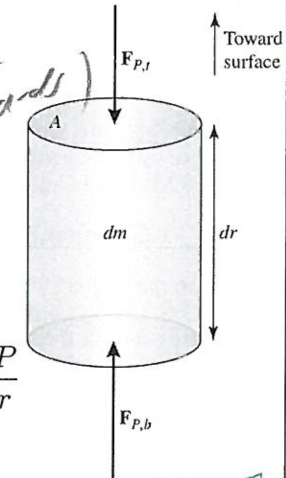
- Light comes only from *surface* of stars, so no good direct way of observing *interior.* Also – neutrinos, oscillations
- Current understanding comes mainly from detailed computer modeling.

• Derivation of Hydrostatic Equilibrium Eq.

- F = ma for mass dm shown: $dm\dfrac{d^2r}{dt^2} = F_g + F_{P,t} + F_{P,b}$ (>0 if outwards)
- F_g = (inwards) force of gravity (M_r = mass inside sphere of radius *r)* 10.2) $F_g = -G\dfrac{M_r dm}{r^2}$
- $F_{P,t}$ & $F_{P,b}$ are pressure forces (F = PA) on top & bottom of cylinder. Net outwards pressure force is $F_{P,b} + F_{P,t} = -A\,dP = -A\dfrac{dP}{dr}dr$
- Using $dm = \rho A\,dr$ & dividing by *A dr* gives (10.5) $\rho\dfrac{d^2r}{dt^2} = -G\dfrac{M_r\rho}{r^2} - \dfrac{dP}{dr}$
- Finally, for a static star we have the eq. of hydrostatic equilibrium (10.6):

$$\frac{dP}{dr} = -G\frac{M_r\rho}{r^2} = -\rho g$$

- The pressure gradient balances gravity.



F10-1 p285

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Example 10.1.1 Estimate pressure at center of Sun, using $M_r = 1\ M_\odot$, $r = 1\ R_\odot$, & average density $\rho = \rho_\odot = 1410$ kg m^{-3}. Assume $P_{surface} = 0$. Eq. 10.6 then gives

know from observable $M_\odot$, $R_\odot$

$$\frac{dP}{dr} \sim \frac{P_s - P_c}{R_s - 0} \sim -\frac{P_c}{R_\odot} \sim -G\frac{M_\odot \rho_\odot}{R_\odot^2} \Rightarrow P_c \sim G\frac{M_\odot \rho_\odot}{R_\odot}$$

$$= 6.673 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}} \frac{1.99 \times 10^{30}\ \mathrm{kg}\ 1410\ \mathrm{kg\,m^{-3}}}{6.955 \times 10^8\ \mathrm{m}} \sim 2.7 \times 10^{14}\ \mathrm{N\,m^{-2}}$$

For better accuracy would need to integrate eq. (10.6),

$$\int_{P_s}^{P_c} dP = P_c = -\int_{R_s}^{R_c} \frac{GM_r\rho}{r^2} dr$$

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but functional forms of $M_r(r)$ & $\rho(r)$ unknown. More accurate calculations give $\cancel{\rho}_c$ P ✓ $= 2.34 \times 10^{16}$ N m^{-2} = 2.3×10^{11} atm!

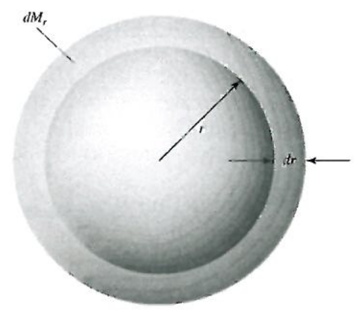
- Equation of Mass Conservation

  - Amount of mass in spherical shell is

$$dM_r = \rho(4\pi r^2 dr)$$

  - Equation of mass conservation:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$



F10.2

p288 §10.2 Pressure Equation of State

To build a stellar model need to know how pressure depends on other properties - need a pressure equation of state (e.o.s.).

Ex. Ideal gas law $PV = NkT$ (V=volume, N=# particles, k= Boltzmann's const., T=absolute temperature).

or $PV = nRT$, but we like to use $n = \frac{N}{V}$ not $n = \frac{N}{N_A}$.

p289 Derivation of Pressure Integral

(Fig. 10.3 p289)

Elastic scattering of particle off wall ⇒ impulse to wall

$\vec{f}\Delta t = -\Delta\vec{p} = 2p_x \hat{i}$

Time between collisions w/ same wall $\Delta t = \frac{2\Delta x}{v_x} \Rightarrow f = \frac{p_x v_x}{\Delta x}$

For isotropic velocity distribution, $\overline{v^2} = \overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} = 3\overline{v_x^2} \Rightarrow$

$\overline{p_x v_x} = m\overline{v_x^2} = \frac{m}{3}\overline{v^2} = \frac{1}{3}pv$ ← why $\frac{1}{3}$?

If $N_p dp$ is the number of particles in momentum range dp, then average force exerted by particles in dp is

$dF(p) = f(p) N_p dp = \frac{1}{3}\frac{N_p}{\Delta x} pv\,dp$ + the total force by all particles is

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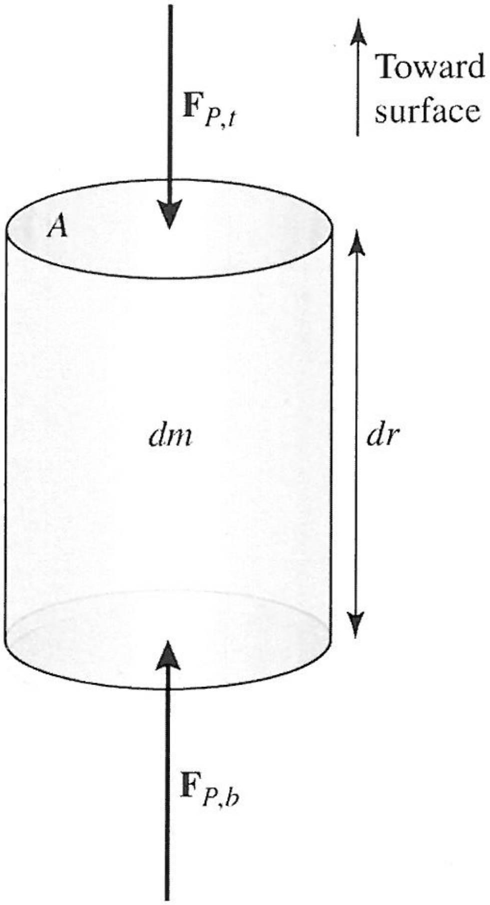

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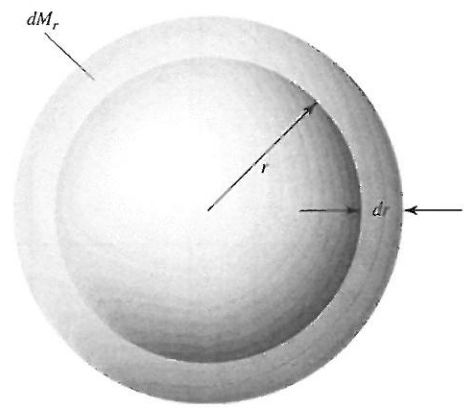

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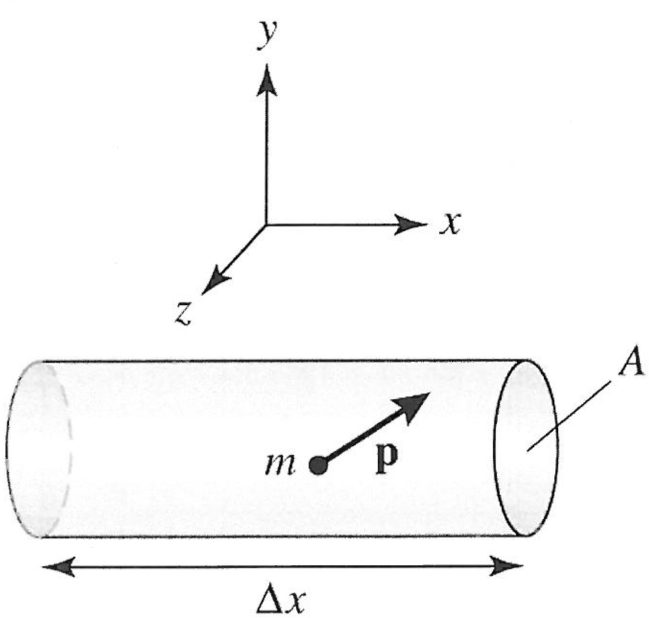
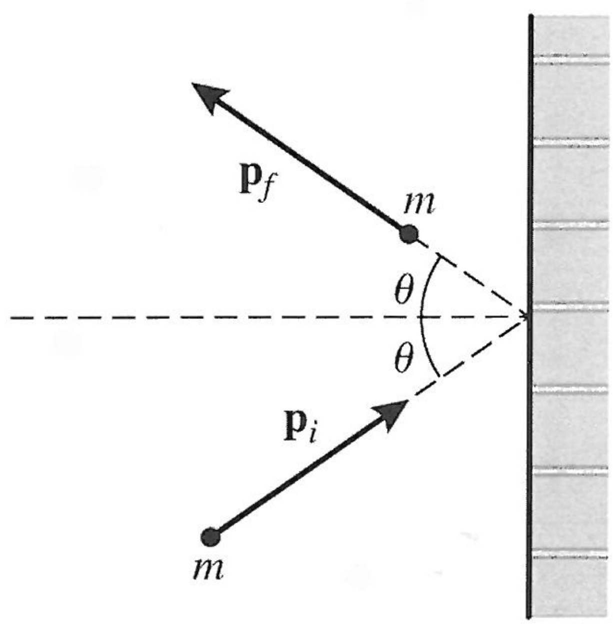


(a)



(b)

F10.3 p. 289

$$F = \frac{1}{3}\int_0^\infty \frac{N_p}{\Delta x} p v \, dp$$

Use $P = \frac{F}{A}$, $\Delta V = A\Delta x$, + $n_p dp = \frac{N_p}{\Delta V} dp \rightarrow P = \frac{1}{3}\int_0^\infty n_p p v \, dp$ (10.8)

This pressure integral gives the pressure if you know the distribution function n_p.

p.291 Ideal Gas Law in Terms of Mean Molecular Weight

For nonrelativistic massive particles, $p = mv$.

For an ideal gas, $n_v dv$ is the Maxwell-Boltzmann distribution

$$n_v dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 dv$$

Plugging into the pressure integral (Prob. 10.5) gives $P_g = nkT$ $\left(n = \frac{N}{V}\right)$ (10.10)

Use $n = \frac{\rho}{\bar{m}}$ (ρ = mass density, $\bar{m}$ = mean mass per particle)

Define mean molecular weight $\boxed{\mu \equiv \frac{\bar{m}}{m_H}}$, $m_H = 1.674 \times 10^{-27}$ kg

$$\boxed{P_g = \frac{\rho k T}{\mu m_H}}$$

The pressure depends on the degree of ionization, since free electrons contribute to pressure.

For completely neutral gas, $\mu_n = \frac{\sum_j N_j A_j}{\sum N_j}$ w/ $A_j = \frac{m_j}{m_H}$

Using the fact that the electron mass is negligible, for completely ionized gases $\mu_i \simeq \frac{\sum_j N_j A_j}{\sum_j N_j(1+z_j)}$

(since element j has z_j electrons).

In general, the degree of ionization can be determined from the Saha eq. (§8.1).

Finally for stars the composition is often written in terms of the mass fractions X, Y, + Z (§9.2)

$$X = \frac{\text{total mass of H}}{\text{total mass of gas}}$$

$$Y = \frac{\text{total mass of He}}{\text{total mass of gas}}$$

$$Z = \frac{\text{total mass of "metals"}}{\text{total mass of gas}}$$

"Metals" means everything but H + He.

It can be shown that for a neutral gas $\frac{1}{\mu_n} \simeq X + \frac{1}{4}Y + \langle \frac{1}{A} \rangle_n Z$
($\langle 1/A \rangle_n \sim 1/15.5$ for solar abundances),

+ for completely ionized gas $\frac{1}{\mu_i} \simeq 2X + \frac{3}{4}Y + \langle \frac{1+z}{A} \rangle_i Z$
(since each H atom contributes 2 particles: a nucleus + an electron).

Also, $\langle \frac{1+z}{A} \rangle_i \simeq \frac{1}{2}$

p. 294 Average Kinetic Energy Per Particle

From 10.8 + 10.10 we have $P = \frac{1}{3}\int_0^\infty m n_v v^2 dv = nkT$

so $\frac{1}{n}\int_0^\infty \frac{1}{2} m v^2 n_v \, dv = \frac{1}{2} m \overline{v^2} = \frac{3}{2} kT$

This is the law of equipartition of energy, from thermodynamics.
The average kinetic energy per particle is $\frac{1}{2}kT$ per degree of freedom.

p. 294 Fermi-Dirac + Bose-Einstein Statistics

There are 2 limits to this analysis.
The velocity integral went to ∞, even though relativity limits us to $v < c$. So this is inaccurate at high temperatures.
At extremely low temperatures or high densities, particles are near their quantum mechanical ground states so they act differently depending on whether they are fermions or bosons.

Fermions (neutron, proton, electron, or any object with half-integer spin $\frac{1}{2}\hbar$, $\frac{3}{2}\hbar$ etc) obey the Pauli exclusion principle — no 2 identical fermions can occupy the same state.
They follow the Fermi-Dirac distribution function.

Bosons (photons, He nuclei, anything with integer spin) follow Bose-Einstein distribution function, + tend to have large numbers of particles occupying the same state.

p. 295 - The Contribution Due to Radiation Pressure

Photons have momentum $p_\gamma = h\nu/c$, so exert pressure.

In the pressure integral 10.8 put $v = c$, $p = h\nu/c$, + $n_p dp = n_\nu d\nu$ →

$$P_{rad} = \frac{1}{3}\int_0^\infty n_\nu \frac{h\nu}{c} c\, d\nu = \frac{1}{3}\int_0^\infty h\nu\, n_\nu d\nu = \frac{1}{3}(\text{energy density}) = \frac{1}{3}aT^4 \quad (9.7)$$

a = radiation constant $= 7.566\times10^{-16}\ J\ m^{-3}\ K^{-4}$

For ideal gas with radiation, $\boxed{P_t = \frac{\rho kT}{\mu m_H} + \frac{1}{3}aT^4}$

Ex. 10.2.1 p295 Use the estimate $P_c = 2.7\times10^{14}\ N m^{-2}$ for center of Sun, estimate T_c.

Radiation pressure is relatively small (we will show), so

$$T_c = \frac{P_c \mu m_H}{\rho k} = \frac{2.7\times10^{14} Nm^{-2}(0.62)1.673\times10^{-27}kg}{1410\ kg\ m^{-3}(1.38\times10^{-23}\ J/K)} = \boxed{1.44\times10^7 K}$$

$$P_{rad} = \frac{1}{3}(7.566\times10^{-16} J\ m^{-3} K^{-4})(1.44\times10^7 K)^4 = \boxed{1.08\times10^{13} Nm^{-2} \ll P_{gas}}$$

p.4 2×10^{16}

p. 296 §10.3 Stellar Energy Sources

In this section we will show that the Sun could not have shone for billions of years if it were powered by gravitational or chemical energy. We will also discuss nuclear fusion in detail.

Gravitation + the Kelvin-Helmholtz Timescale

How long could the Sun burn w/ $L_\odot$ w/ grav. energy source?

1st calculate U = gravitational P.E.

U(2 point masses) = W(to bring them from ∞ to r) $= -\frac{GMm}{r}$

Since F due to spherical mass (M_r, r) is same as point mass,

dU(additional spherical shell $dm = 4\pi r^2 \rho\, dr$) $= -\frac{G M_r 4\pi r^2 \rho}{r} dr$

$$U_g = -4\pi G\int_0^R M_r \rho r\, dr$$

Need to know $\rho(r)$, but approximate by uniform sphere $\rho \sim \bar\rho = \frac{M}{\frac{4}{3}\pi R^3}$

$$U_g = -4\pi G\int_0^R \left(\frac{M}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3\right)\frac{M}{\frac{4}{3}\pi R^3} r\, dr = -\frac{3}{5}\frac{GM^2}{R}$$

According to the virial theorem (§2.4), the time-averages of the total + potential energies $\langle E\rangle$ + $\langle U\rangle$ satisfy $\langle E\rangle = \frac{1}{2}\langle U\rangle$

(example: circular gravitational orbit)

So $E \sim -\frac{3}{10}\frac{GM^2}{R}$

Ex 10.3.1 p.297 If Sun were originally much larger, how much E would be liberated in collapsing to $R_\odot$? How long could it shine at $L_\odot$?

$$R_i >> R_\odot \Rightarrow E_i - E_f = -\frac{3}{10}\frac{GM_\odot^2}{R_i} + \frac{3}{10}\frac{GM_\odot^2}{R_\odot} \sim \frac{3}{10}\frac{GM_\odot^2}{R_\odot}$$

$$= \frac{3}{10}\frac{6.673\times10^{-11}(1.989\times10^{30})^2}{6.955\times10^{8}} \sim 1.1\times10^{41}\,J$$

Kelvin-Helmholtz timescale $t_{KH} = \frac{|\Delta E|}{L_\odot} \sim \frac{1.1\times10^{41}\,J}{3.840\times10^{26}\,W} \sim 3.0\times10^{14}\,s \sim 10^7\,yr$

But we know that Earth + Moon are $\sim 4.5\times10^9$ yr old (Moon rocks $\sim 4\times10^9$ yr).

chemical energy? energy/atom $\sim$ 1–10 eV

Prob. 10.3 $\Rightarrow$ the amount of available chemical energy is way too low.

The Nuclear Timescale

Nuclear reactions can liberate MeV/nucleus.

Element specified by Z = # of protons

Isotope specified by N = # of neutrons

A = N + Z = # of nucleons = mass number

$m_p = 1.673\times10^{-27}$ kg $= 1.0073\,u$

$m_n = 1.675\times10^{-27}$ kg $= 1.0087\,u$

$m_e = 9.109\times10^{-31}$ kg $= 0.00055\,u$

$1\,u = \frac{1}{12}$ (mass of C-12 atom) $= 1.661\times10^{-27}$ kg $= 931.5$ MeV/c^2

Fusion = combining small nuclei to make larger ones (4H → He)

Fission = splitting large nuclei (U)

$M(2p+2n) - M(He) = 0.0287\,u = 26.73$ MeV/c^2 $= 0.7\%$ of total mass

26.73 MeV = binding energy of He

Ex. 10.3.2 p.299 Is this a reasonable energy source?

Assume Sun originally 100% H but only inner 10% hot enough to fuse.

$$E_{nuclear} = 0.1\times0.007\times M_\odot c^2 = 1.3\times10^{44}\,J$$

$$t_{nuclear} = \frac{E_{nuclear}}{L_\odot} \sim 10^{10}\,yr \quad \text{- fine!}$$

p. 300 Quantum Mechanical Tunneling

Can nuclear fusion actually occur inside a star?

The potential energy $U(r)$ between 2 p's is Coulomb repulsion + short range strong nuclear attraction. (F 10.4 p. 300).

W/O QM effects, T must be high enough for p's to go over potential barrier,

$$\frac{1}{2}\mu_m \overline{v^2} = \frac{3}{2} k T_{classical} = \frac{1}{4\pi\epsilon_0}\frac{Z_1 Z_2 e^2}{r}$$

μ_m = reduced mass $= \frac{m_1 m_2}{m_1+m_2}$ (eq. 2.22)

r = distance of closest approach ~ 1 fm $= 10^{-15}$ m (nuclear size)

$Z_1 = Z_2 = 1$, $m_1 = m_2 = m_p \Rightarrow T_{classical} \sim 10^{10}$ K >> $T_{central} = 1.57 \times 10^7$ K

QM tunneling (PHY 1371, 3080):

rough estimate: proton must be within 1 deBroglie wavelength $\lambda = h/p$ to fuse $\Rightarrow T_{quantum} \sim 10^7$ K — ok.

(this also explains strong T dependence of reaction rates!)

p. 302 Nuclear Reaction Rates + the Gamow Peak

Nuclear reaction probabilities are expressed in terms of cross sections, which are calculated quantum mechanically or measured experimentally.

$$\sigma(E) = \frac{\#\text{ of reactions/nucleus/time}}{\#\text{ of incident particles/area/time}} \quad (\text{units} = \frac{\text{area}}{\text{nucleus}})$$

(oh F 10.5 p. 303)

# of reactions in dt due to particles in energy range dE = # of these particles in cylinder of area $\sigma(E)$ + length $v(E)dt$.

density of these particles comes from Maxwell-Boltzmann:

(10.28) $$n_E\, dE = \frac{2n}{\pi^{1/2}}\frac{1}{(kT)^{3/2}} E^{1/2} e^{-E/kT} dE$$ (proved in prob. 10.6)

Usually $\sigma(E)$ has the form $\sigma(E) = \frac{S(E)}{E} e^{-bE^{-1/2}}$ where $S(E)$ is a slowly-varying function + the exponential comes from the probability of tunneling thru the Coulomb barrier.

Putting it all together, the reaction rate for incident particles i hitting target particles x is

$$r_{ix} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_x}{(\mu_m \pi)^{1/2}} \int_0^\infty S(E) e^{-bE^{-1/2}} e^{-E/kT} dE \quad (10.33)$$

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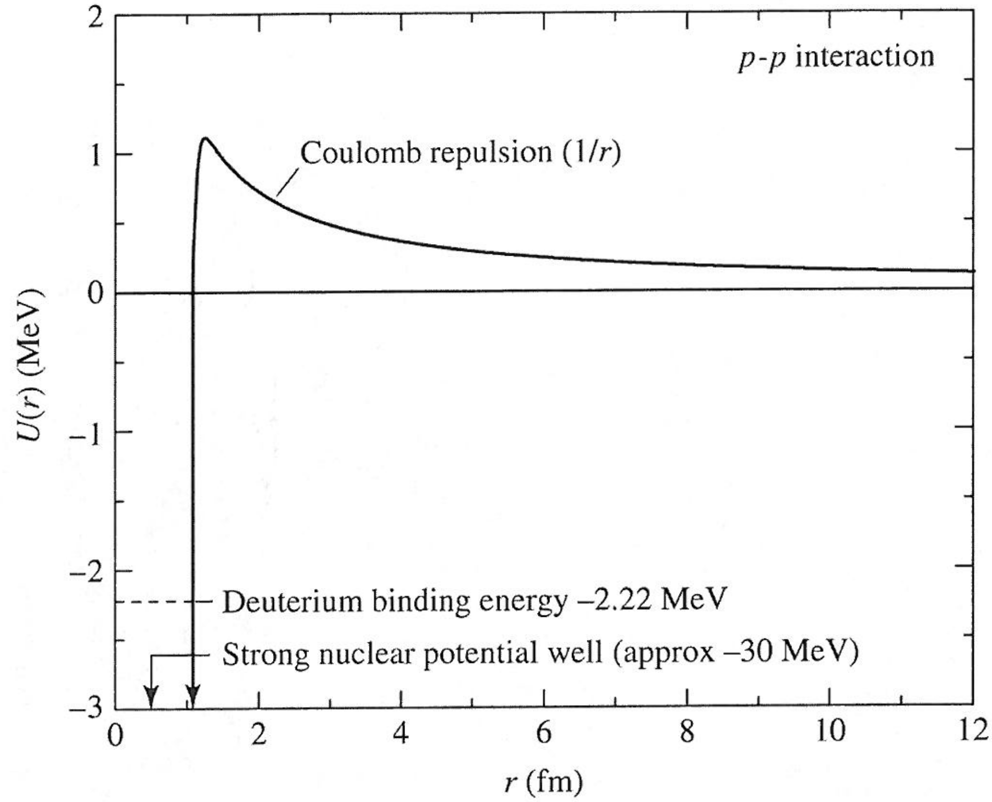
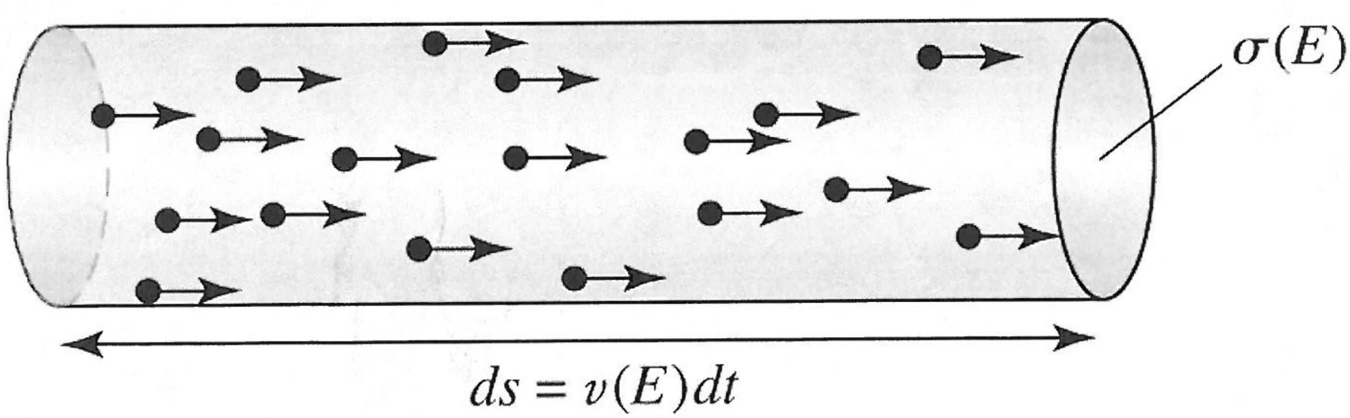


F10.4 p.300



F10.5 p. 303

(Oh F10.6 p.305) Assuming $S(E)$ is slowly varying, the reaction rate is mainly due to energies near Gamow peak (George Gamow 1904–68), which occurs at $E_0 = \left(\frac{bkT}{2}\right)^{2/3}$ (eq. 10.34) (Prob. 10.9)

p.306 <u>Resonance</u> Sometimes $S(E)$ is not slowly varying
(Oh F10.7 p.306) This is due to resonance between energy of incoming particle + differences between energy levels in target nucleus.

<u>Electron screening</u> Free electrons group around positive nuclei, effectively reducing the Coulomb repulsion + increasing helium production rate by 10–50%.

p.307 <u>Representing Nuclear Reaction Rates Using Power Laws</u>

The energy production rate can often be approximated by

$$\epsilon_{ix} = \epsilon_0' X_i X_x \rho^\alpha T^\beta \quad (W/kg) \qquad (10.35)$$

where ϵ_0' is a constant, X_i + X_x are the mass fractions of the two species, ρ is the mass density, $\alpha \approx 1$ for binary reactions, + $\beta \sim 1-40$.

Adding up the ϵ_{ix}'s for all species gives the total rate.

<u>p.307 the Luminosity Gradient Equation</u>

Total energy released per second in shell of mass $dm = 4\pi r^2 \rho dr$ is $\epsilon\, dm$ ($\epsilon = \epsilon_{nuclear} + \epsilon_{gravity}$), + must = dL = luminosity leaving shell at $r+dr$ − luminosity entering shell at r $\Rightarrow$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon \qquad (10.36)$$

(another fundamental stellar structure eq.)

p.308 <u>Stellar Nucleosynthesis + Conservation Laws</u>

H burning results in $4H \rightarrow {}^4_2He$, but it is <u>extremely</u> unlikely that 4 (or even 3) particles can be together to interact – all reactions are actually <u>binary</u>.

In every reaction, electric charge, nucleon #, + lepton # are conserved. Electrons + neutrinos have lepton # 1, their antiparticles have −1. Neutrinos ν have $q=0$, $m_\nu < 2.2\, eV/c^2$, $\sigma_\nu \sim 10^{-48} m^2 \Rightarrow$ mfp ~ 10pc at normal stellar densities.

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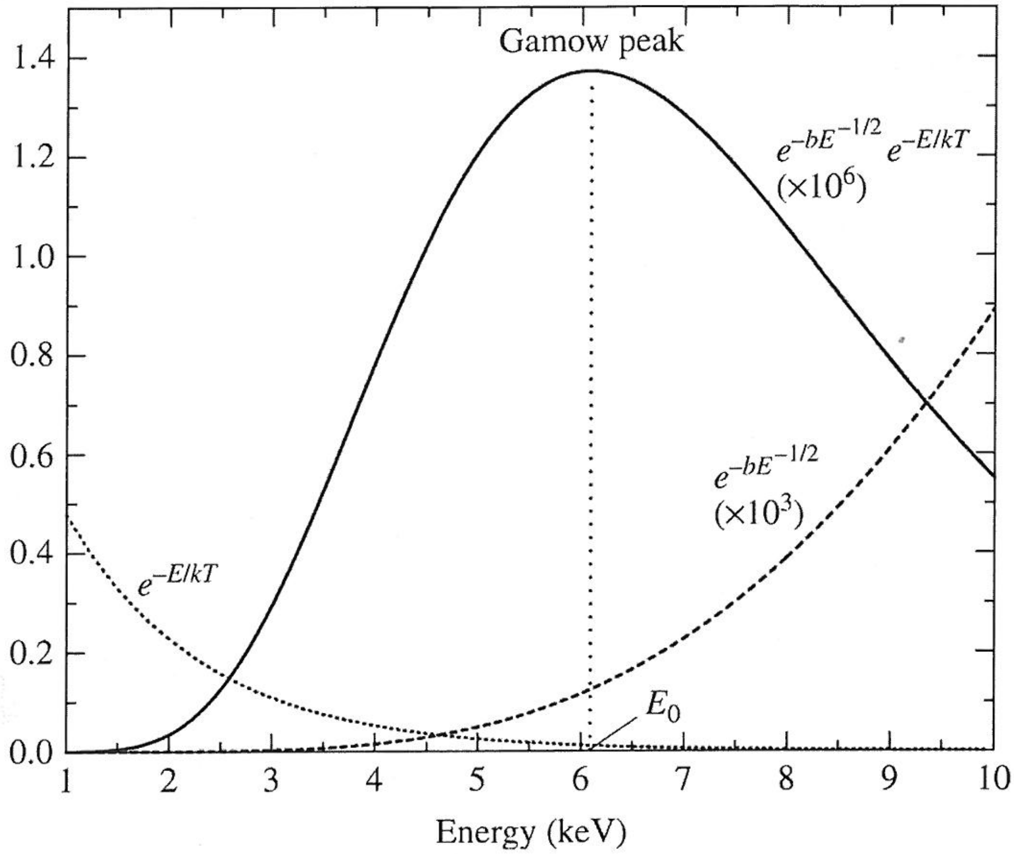
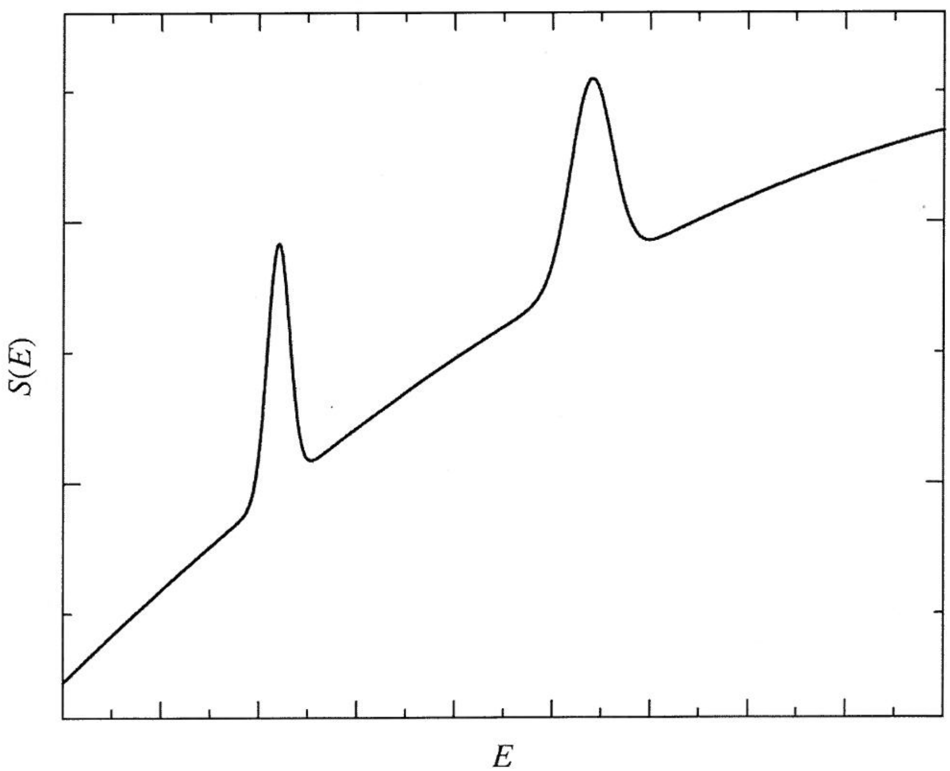


F10.6 p.305



F10.7 p. 306

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Notation $^A_Z X$, A = nucleon #, Z = atomic #, X = chemical symbol

p.309 The Proton-Proton Chains

The 1st proton-proton chain (PP I) has the net effect

$$4\,^1_1H \rightarrow \,^4_2He + 2e^+ + 2\nu_e + 2\gamma$$

The actual steps are

$$^1_1H + \,^1_1H \rightarrow \,^2_1H + e^+ + \nu_e$$
$$^2_1H + \,^1_1H \rightarrow \,^3_2He + \gamma$$
$$^3_2He + \,^3_2He \rightarrow \,^4_2He + 2\,^1_1H$$

The 1st step is the slowest, since it involves $p \rightarrow n + e^+ + \nu_e$ — via the weak force.

^3_2He can interact w/ ^4_2He instead of ^3_2He, in PP II chain.

^7_4Be can be converted to ^4_2He in the PP III chain.

(ch F10.8 p.310) This figure shows all 3 chains, along w/ branching ratios appropriate to core of Sun.

The net energy production rate is approximately

$\epsilon_{pp} \simeq \epsilon'_{0,pp}\, \rho X^2 f_{pp} \psi_{pp} C_{pp} T_6^4$ where

$\epsilon'_{0,pp} = 1.08 \times 10^{-12}\ W\,m^3\,kg^{-2}$, X = H mass fraction; f, ψ, & C are factors of size ~1, & $T_6 = T/10^6 K$.

p.311 the CNO Cycle

Proposed by Hans Bethe (1906-2005) in 1938.

C, N, O used as catalysts to convert H to He (reactions in text).

$$\epsilon_{CNO} \simeq \epsilon_{0,CNO}\, \rho X X_{CNO} T_6^{19.9}$$

$T_6^{19.9} \Rightarrow$ CNO more important in more massive stars, which burn hotter.

$4H \rightarrow \,^4_2He$ reduces # of particles $\Rightarrow$ pressure $\Rightarrow$ core contracts & heats up, allowing He to burn.

p312 Triple Alpha Process of He Burning

$^4_2He + \,^4_2He \rightleftharpoons \,^8_4Be$ (reverse happens immediately unless...

$^8_4Be + \,^4_2He \rightarrow \,^{12}_6C + \gamma$

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$${}^1_1H + {}^1_1H \rightarrow {}^2_1H + e^+ + \nu_e$$

$${}^2_1H + {}^1_1H \rightarrow {}^3_2He + \gamma$$

pp I

69% 31%

pp II

pp III

$${}^3_2He + {}^3_2He \rightarrow {}^4_2He + 2\,{}^1_1H$$

(PP I)

$${}^3_2He + {}^4_2He \rightarrow {}^7_4Be + \gamma$$

99.7% 0.3%

$${}^7_4Be + e^- \rightarrow {}^7_3Li + \nu_e$$

$${}^7_3Li + {}^1_1H \rightarrow 2\,{}^4_2He$$

(PP II)

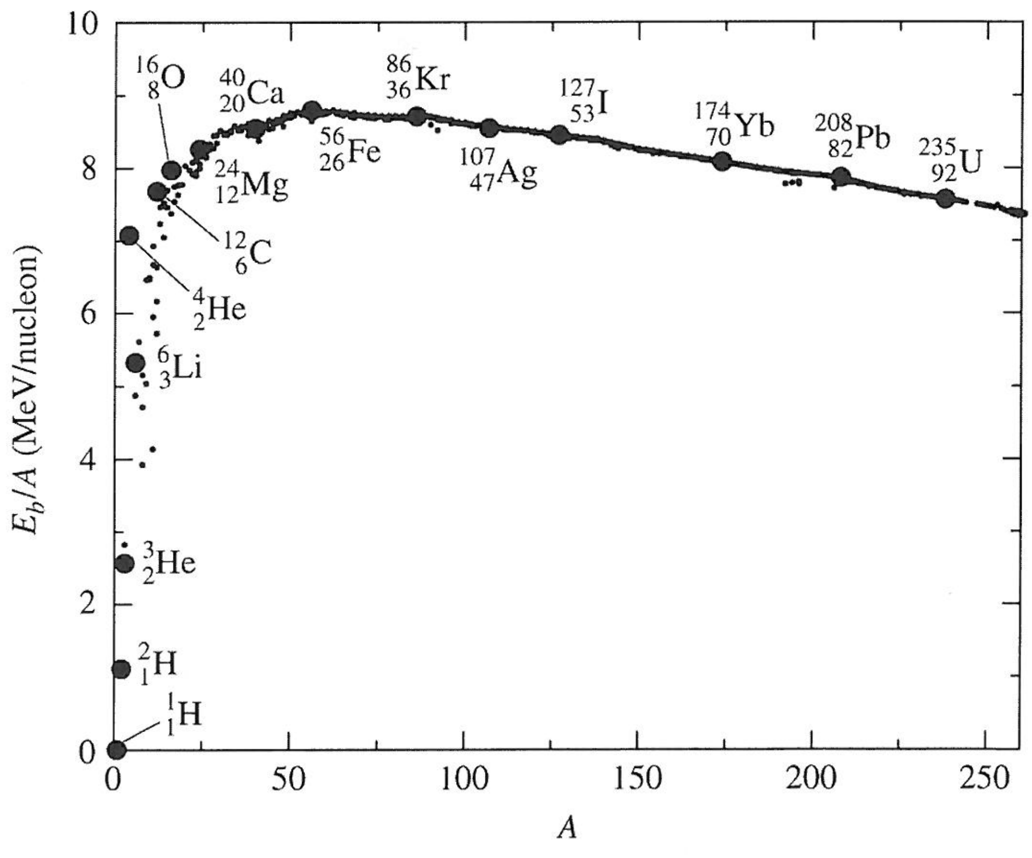
$${}^7_4Be + {}^1_1H \rightarrow {}^8_5B + \gamma$$

$${}^8_5B \rightarrow {}^8_4Be + e^+ + \nu_e$$

$${}^8_4Be \rightarrow 2\,{}^4_2He$$

(PP III)

F10.8 p.310



F10.9 p. 314

p.313 Carbon + Oxygen Burning

when enough ${}_6C$ has been produced, it can react to form ${}_8O$, ${}_{10}Ne$, ${}_{11}Na$, ${}_{12}Mg$, ${}_{14}Si$, ${}_{15}P$, + ${}_{16}S$

p.314 Binding Energy per Nucleon

$E_b = \Delta mc^2 = [Zm_p + (A-Z)m_n - m_{nucleus}]c^2$

(ch F10.9 p.314) E_b/A

Stars can produce energy by burning up to ${}^{56}_{26}Fe$, the most stable nucleus.
Most abundant species in cosmos, in order: ${}^1_1H$, ${}^4_2He$, ${}^{16}_8O$, ${}^{12}_6C$...
nuclei w/ high E_b/A.
(Elements above Fe produce energy by fission, not fusion.)

p315 §10.4 Energy Transport + Thermodynamics

The last remaining stellar structure eq. is that of energy transport.
3 energy transport mechanisms: radiation, convection, conduction
Conduction usually unimportant in stars - ignore.

p.316 Radiative Temperature Gradient

We need the concept of opacity, developed in §9.2.

$dI = -\kappa\rho I\, ds$ (9.13) I = intensity of beam, κ (kappa) = opacity ($m^2\, kg^{-1}$), ρ = mass density, ds = distance, dI is the amount of radiation removed from the beam by absorption or scattering.

$\bar{\kappa}$ = Rosseland mean opacity (frequency averaged).

In spherical coords, + using the radiant flux F_{rad} ($W m^{-2}$) gives

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c} F_{rad}$$

Also $P_{rad} = \frac{1}{3}aT^4 \Rightarrow \frac{dP_{rad}}{dr} = \frac{4}{3}aT^3\frac{dT}{dr}$

From Ch.3 $F_{rad} = \frac{L_r}{4\pi r^2}$ where L_r is the total luminosity coming out of a sphere of radius r.

Putting it all together gives $\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2}$ (10.68)

Pressure Scale Height

Convection governed by complicated set of 3D Navier-Stokes equations.
Convection is 3D, but due to computational limitations models mainly 1D.

Convection length can be $\sim R_*$, + convection time $\sim t_{dynamical}$.
But we'll do what we can.
Convection mixing length is often comparable to pressure scale height $H_p \equiv -\frac{P}{dP/dr} = \frac{P}{\rho g}$ (10,70)

which is the radial distance over which P decreases by $\frac{1}{e}$.
Example 10.4.1 p. 317 $\Rightarrow$ In Sun, $H_p \sim R_\odot/10$.

p. 317 Internal Energy + 1st Law of Thermodynamics

$dU = đQ - đW$

U = internal energy = state variable
$đQ$ = heat flow in, $đW$ = work done by fluid element
$đ \Rightarrow$ inexact differential (process, so path dependent).
Ideal monatomic gas $\Rightarrow U = \frac{3}{2}\frac{kT}{\mu m_H}$ (internal energy / mass)

p. 318 Specific Heats

Well-known formulae from thermodynamics:

$C_p \equiv \frac{\partial Q}{\partial T}\Big|_P \quad C_V \equiv \frac{\partial Q}{\partial T}\Big|_V \quad đW = PdV$

$dU = C_V dT \quad C_p = C_V + nR \;(n = \text{moles/kg}) \quad \gamma \equiv \frac{C_p}{C_V}$

$PV^\gamma = K = \text{constant}$ (for adiabatic processes) $\Rightarrow P = K' T^{\gamma/(\gamma-1)}$

fast p. 321 Adiabatic Sound Speed

Sound waves usually oscillate fast enough that $đQ \approx 0 \Rightarrow$ adiabatic
$\Rightarrow$ sound speed $v_s = \sqrt{B/\rho} = \sqrt{\gamma P/\rho}$

B = bulk modulus $= -V\frac{\partial P}{\partial V}\Big|_{ad} = -V\frac{\partial}{\partial V}(KV^{-\gamma}) = \gamma V K V^{-\gamma-1} = \gamma P$

fast Example 10.4.2 p321 Sun: monatomic gas $\Rightarrow \gamma = \frac{C_p}{C_V} = \frac{\frac{5}{2}nR}{\frac{3}{2}nR} = \frac{5}{3}$

Use $\bar{P} = P_c/2$, $\bar{\rho}_\odot \Rightarrow \bar{v}_s \approx \left(\frac{5}{3}\frac{\bar{P}}{\bar{\rho}_\odot}\right)^{\frac{1}{2}} \approx 4 \times 10^5 \text{ m s}^{-1}$

Time to traverse $R_\odot$ = $t \approx R_\odot/\bar{v}_s \approx 29$ minutes

Adiabatic Temperature Gradient

$P = \frac{\rho k T}{\mu m_H} \Rightarrow \frac{dP}{dr} = -\frac{P}{\mu}\frac{d\mu}{dr} + \frac{P}{\rho}\frac{d\rho}{dr} + \frac{P}{T}\frac{dT}{dr}$

$P = K\rho^\gamma \Rightarrow \frac{dP}{dr} = \gamma\frac{P}{\rho}\frac{d\rho}{dr}$

Combining these + assuming $\mu \approx$ const. $\Rightarrow$

$$\frac{dP}{dr} = \frac{1}{\gamma}\frac{dP}{dr} + \frac{P}{T}\frac{dT}{dr} \Rightarrow \left.\frac{dT}{dr}\right|_{ad} = \left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} = -\left(1-\frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2}$$

Using $g = \frac{GM_r}{r^2}$, $1-\frac{1}{\gamma} = \frac{\gamma-1}{\gamma} = \frac{C_p/C_v - 1}{C_p/C_v} = \frac{C_p - C_v}{C_p} = \frac{nR}{C_p}$, + $\frac{k}{\mu m_H} = nR$ gives

$$\left.\frac{dT}{dr}\right|_{ad} = -\frac{g}{C_p}$$

This describes T in a bubble as it rises + expands adiabatically.
We will show that if the actual T deep inside a star decreases faster than this: $\left|\frac{dT}{dr}\right|_{act} > \left|\frac{dT}{dr}\right|_{ad}$ it is superadiabatic, + energy transport is primarily by convection (near surface both radiation + convection can be significant).

p322 A Criterion for Stellar Convection
In this section they prove that if $\left|\frac{dT}{dr}\right|_{act} > \left|\frac{dT}{dr}\right|_{ad}$, the buoyant force on a bubble increases as it rises (assuming μ = const.), + convection will result.
They also show that this can be written $\frac{T}{P}\frac{dP}{dT} = \frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma-1}$

p.325 The Mixing-Length Theory of Superadiabatic Convection
Let ℓ = mixing length = distance bubble rises before dissipating + giving up its heat, + assume $\ell = \alpha H_p$ with α = free parameter.
(Numerical models $\Rightarrow 0.5 < \alpha < 3$).
$\delta T = T_{bubble} - T_{surroundings}$ after it has risen by ℓ.
$P_b = P_s$, so the heat that flows out of the bubble per volume $= \delta q = C_p \delta T \rho$.
Convective energy flux $F_c = \delta q \bar{v}_c = (C_p \delta T)\rho \bar{v}_c$ $\left(\frac{J}{kg}\cdot\frac{kg}{m^3}\cdot\frac{m}{s} = \frac{J}{m^2 s}\right)$
where $\bar{v}_c$ = average velocity of convective bubble.
$\bar{v}_c$ is found by equating the work (per mass) $\langle f_{net}\rangle \ell$ to the kinetic energy $\frac{1}{2}\rho v^2$ + putting in another fudge factor β $(0<\beta<1)$

$$\bar{v}_c = \left(\frac{2\beta\langle f_{net}\rangle \ell}{\rho}\right)^{\frac{1}{2}}$$

f_{net} = buoyant force − gravity

After some manipulation we find

$$F_c = \rho C_p \left(\frac{k}{\mu m_H}\right)^2 \left(\frac{T}{g}\right)^{\frac{3}{2}} \beta^{\frac{1}{2}} \left[\delta\left(\frac{dT}{dr}\right)\right]^{\frac{3}{2}} \alpha^2 \quad (10.99)$$

where $\delta\left(\frac{dT}{dr}\right) = \left.\frac{dT}{dr}\right|_{ad} - \left.\frac{dT}{dr}\right|_{act}$.

If all the flux is carried by convection: $F_c = \frac{L_r}{4\pi r^2} \Rightarrow$

$$\frac{\delta\left(\frac{dT}{dr}\right)}{|dT/dr|_{ad}} = \left(\frac{L_r}{4\pi r^2}\right)^{2/3} C_p^{-\frac{1}{3}} \rho^{-\frac{2}{3}} \alpha^{-4/3} \left(\frac{\mu m_H}{k}\right)^{4/3} \frac{1}{T} \beta^{-\frac{1}{3}}$$

Example 10.4.3 p.328 For Sun's convection zone, assume $\alpha = 1, \beta = \frac{1}{2} \Rightarrow$

$\frac{\delta(dT/dr)}{|dT/dr|_{ad}} \sim 4.1 \times 10^{-7}$, so we don't need to be very superadiabatic

$\bar{v}_c \sim 50 \frac{m}{s} \sim 10^{-4} v_s$

Mixing length theory is adequate for many problems, but the real solution is probably 3D (+ time) numerical calculations

p.329 §10.5 Stellar Model Building

p.330 Summary of Equations of Stellar Structure

$$\frac{dP}{dr} = -G\frac{M_r \rho}{r^2} \quad (10.6) \qquad \frac{dM_r}{dr} = 4\pi r^2 \rho \quad (10.7)$$

$$\frac{dL_r}{dr} = 4\pi r^2 \rho \epsilon \quad (10.36)$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2} \quad \text{(radiation) (10.68)}$$

$$= -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} \quad \text{(adiabatic convection) (10.89)}$$

Entropy - skip

p.331 The Constitutive Relations

These depend on the physical properties of the stellar material. They are pressure P, opacity $\bar{\kappa}$, + energy production rate ϵ as functions of ρ, T, + composition.

$\bar{\kappa}$ usually presented in tabular form + interpolated.

ϵ requires reaction networks w/ abundances of all isotopes.

p.332 Boundary Conditions

$M_r, L_r \to 0$ as $r \to 0$

Usually use $T, P, \rho \to 0$ as $r \to R_*$

the Vogt-Russell theorem

"the mass + the composition structure throughout a star uniquely determine its radius, luminosity, + internal structure, as well as its subsequent evolution."

You can't also specify a radius + luminosity - when you integrate in from R_* it won't be correct at the center.

p. 333 Numerical Modeling of the Stellar Structure Equations

1D numerical stellar models use shells (F10.11 p.334)

Differential equations are replaced by difference equations:

$$P_{i+1} = P_i + \frac{\Delta P}{\Delta r}\delta r$$

Integrate from R_* inwards, from $r=0$ outwards, or both.

Physical processes at center differ from those at surface, so integrate both directions to some fitting point, + iterate if they don't match.

Simple stellar structure code StatStar described in Appendix L, + can be downloaded from text website.

It uses homogeneous composition, + adiabatic convection, but it still produces reasonable models.

p. 334 Polytropic Models + the Lane-Emden Equation

The mechanical eqs. 10.6+7 $\frac{dP}{dr} = -G\frac{M_r\rho}{r^2}$ + $\frac{dM_r}{dr} = 4\pi r^2\rho$ could be solved if there were a simple relationship between P + ρ (w/o reference to T + composition).

Models with $P = K\rho^\gamma$ are called polytropes.

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \frac{d}{dr}\left(-GM_r\right) = -G\frac{dM_r}{dr} = -4\pi G r^2\rho$$

For $P = K\rho^\gamma$, this is

$$-4\pi G\rho = \frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\cdot K\gamma\rho^{\gamma-1}\frac{d\rho}{dr}\right) = \frac{\gamma K}{r^2}\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right]$$

Let $\gamma = \frac{n+1}{n}$, n = polytropic index

$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left[r^2\rho^{(1-n)/n}\frac{d\rho}{dr}\right] = -4\pi G\rho$$

Express ρ in terms of a scaling factor ρ_c + a dimensionless function $D(r)$: $\rho(r) = \rho_c[D_n(r)]^n$ $\quad 0 \le D_n \le 1$

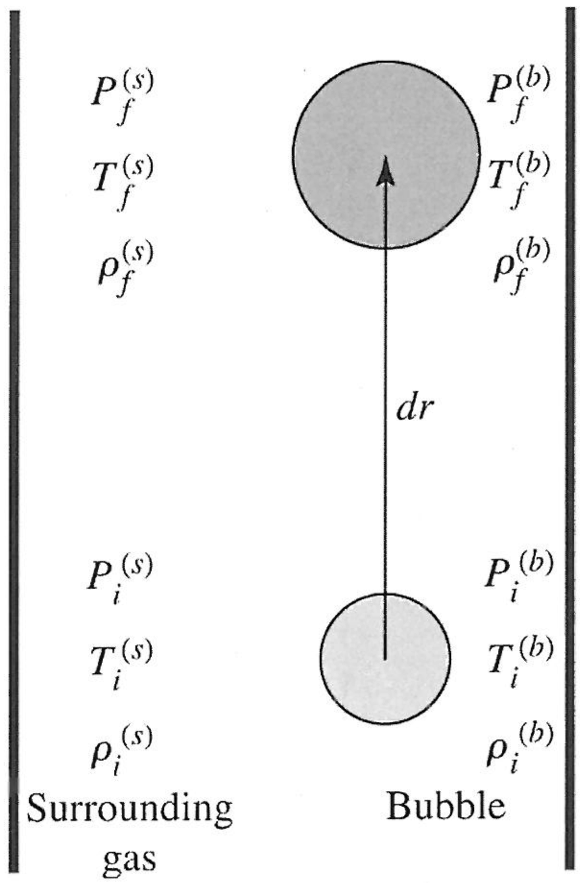
ρ_c turns out to be central density.

Plugging this in + simplifying gives $\left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dD_n}{dr}\right) = -D_n^n$

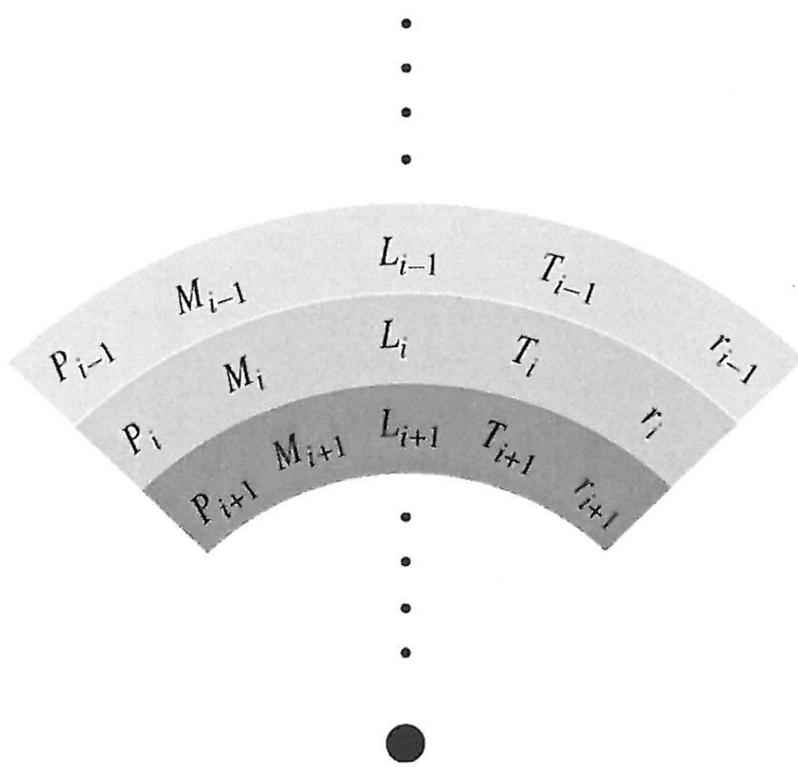
Define $\lambda_n \equiv \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$, which must have length units

16b

1/19/16



F10.10 p. 323



F10.11 p. 334

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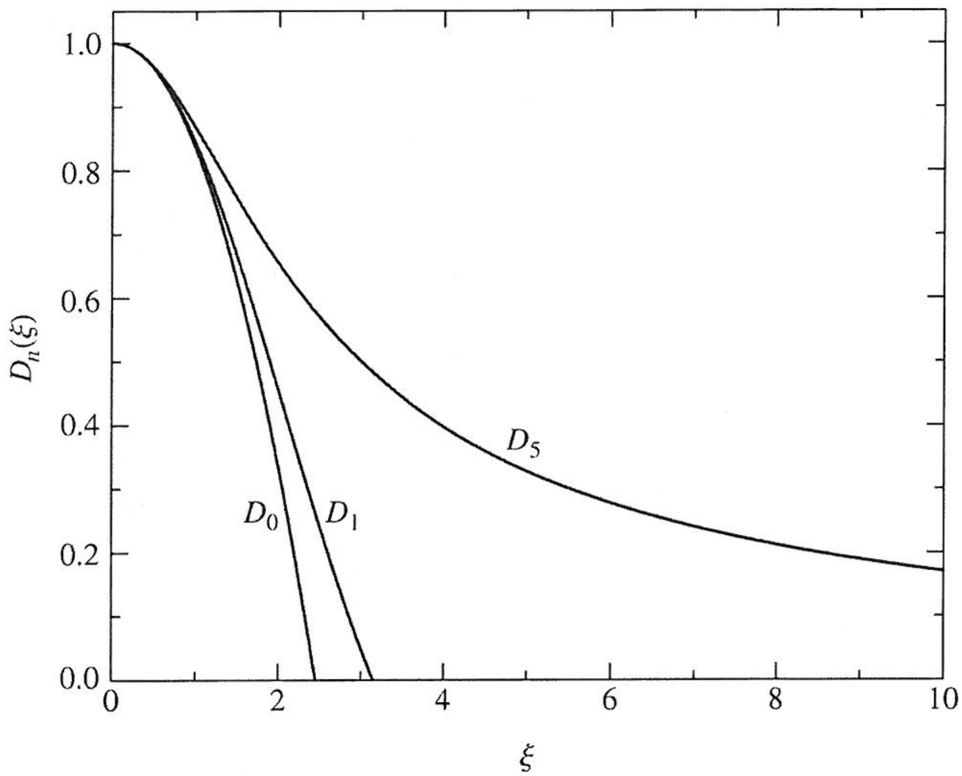
Change to dimensionless variable ξ: $r \equiv \lambda_n \xi$ (ξ = xi: ksee)
This gives $\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2 \frac{dD_n}{d\xi}\right] = -D_n^n$, the "famous"
Lane-Emden eq. (J. Homer Lane 1819-1880, Robert Emden 1862-1940).
Solving gives $D_n(r) \to \rho(r) \to P(r) = K\rho^{(n+1)/n}$, + eos $\Rightarrow T(r)$.
2nd-order diff. eq. $\Rightarrow$ need 2 boundary conditions.
Surface of star: $D_n(\xi_1) = 0$
$\frac{dP}{dr} = -\rho g \to 0$ at $r=0 \Rightarrow \frac{dD_n}{d\xi} = 0$ at $\xi = 0$.
Also normalize w/ $D_n(0) = 1$ so $\rho(\xi=0) = \rho_c$.
It can be shown that the total mass is $M = -4\pi \lambda_n^3 \rho_c \xi_1^2 \left.\frac{dD_n}{d\xi}\right|_{\xi_1}$
There are only 3 analytic solutions to Lane-Emden eq.: $n = 0, 1, 5$
$D_0(\xi) = 1 - \frac{\xi^2}{6}$, $\xi_1 = \sqrt{6}$ (prob. 10.17) (oh F10.12 p339)
$D_1(\xi) = \frac{\sin\xi}{\xi}$, $\xi_1 = \pi$
$D_5(\xi) = [1 + \xi^2/3]^{-1/2}$, $\xi_1 \to \infty$ (but M = finite)
For $n > 5$, $M = \infty$, so must have $0 \le n \le 5$.
Ideal monatomic adiabatic gas, or white dwarf $\Rightarrow \gamma = \frac{5}{3} \Rightarrow n = 1.5$
(must be solved numerically).
They also describe the $n=3$ "Eddington standard model" for radiative equilibrium.
A fully degenerate relativistic white dwarf also has $n=3$.
$n=1.5$ + $n=3$ are the most physically significant polytropes.

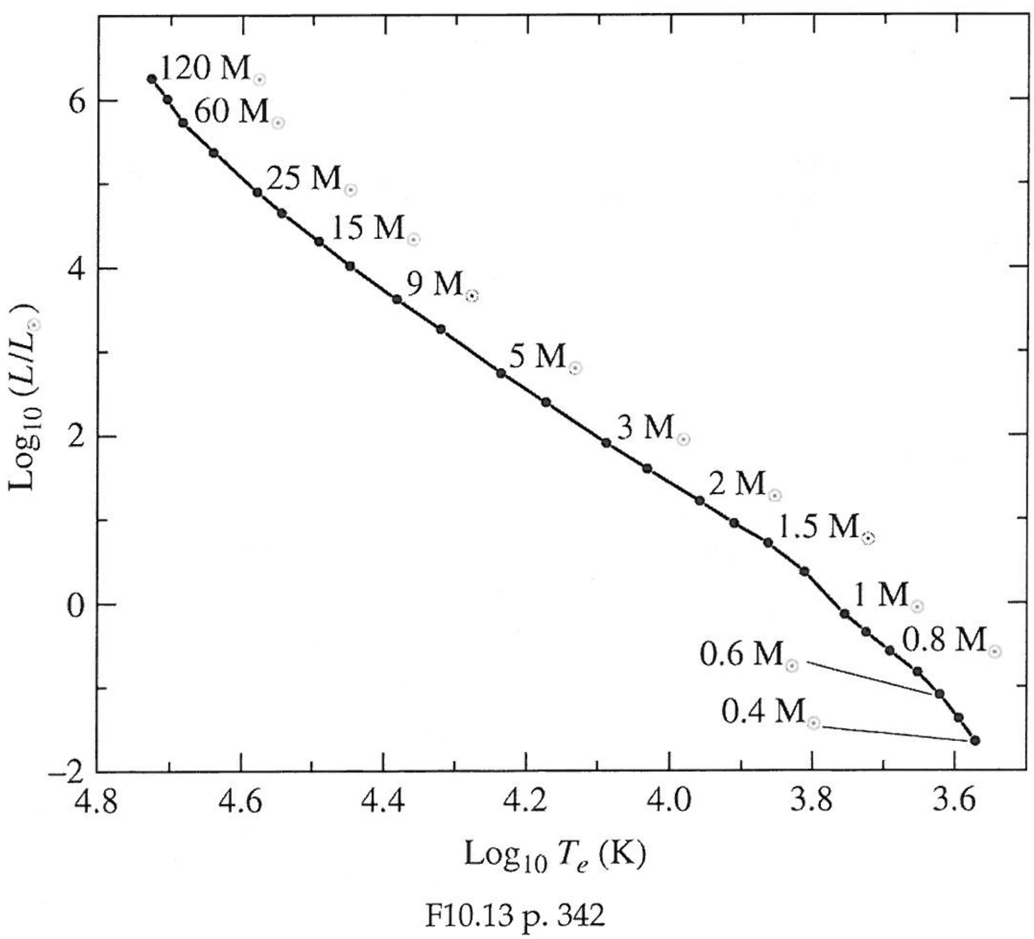
<u>p.340 §10.6 The Main Sequence</u>
Stellar spectra $\Rightarrow$ stellar atmospheres have $X \sim 0.7$, $0 < Z < 0.03$,
+ the rest Y (Helium).
Due to the lower Coulomb barrier, H $\to$ He is the 1st reaction
(via pp chains and/or CNO cycle).
H burning is relatively slow ($\sim 10^{10}$ yr for Sun), so observed changes are slow.
Since most stars have similar composition, structure varies smoothly
w/ mass. (CNO has strong T-dependence, so dominates for
high-mass stars while pp chains dominate for low-mass stars.)
The lowest-mass star for which T_c is high enough to burn H is
$\sim 0.072\,M_\odot$ (for solar composition) or $0.09\,M_\odot$ (for $Z \simeq 0$).
$> 90\,M_\odot \Rightarrow$ thermal oscillations (+, as we will see, extreme mass loss).

17 b

1/21/16



F10.12 p. 339



F10.13 p. 342

p. 341 The Eddington Luminosity Limit

Better analysis than text:

$dI = -\bar{\kappa}\rho I\, ds \Rightarrow dF_{rad} = -\bar{\kappa}\rho\, dr\, F_{rad}$ $\left(\frac{J}{m^2 \cdot s}\right)$ is the radiative energy deposited per area per time.

For radiation, momentum = energy/c $\Rightarrow$ $\frac{\text{momentum}}{\text{area}\cdot\text{time}} = \frac{\text{force}}{\text{area}} = \frac{\bar{\kappa}\rho\, dr\, F_{rad}}{c}$ $\left(\frac{N}{m^2}\right)$

Force on volume $A\,dr = \frac{A\bar{\kappa}\rho\, dr\, F_{rad}}{c}$

Inwards gravitational force on mass $dm = \rho A\, dr = \frac{G M_r \rho A\, dr}{r^2}$

Forces balance when $\frac{A\bar{\kappa}\rho\, dr\, F_{rad}}{c} = \frac{G M_r \rho A\, dr}{r^2} \rightarrow F_{rad} = \frac{G M_r}{r^2}\frac{c}{\bar{\kappa}}$

or when L = Eddington luminosity $L_{Ed} = 4\pi r^2 F_{rad} = \frac{4\pi G c}{\bar{\kappa}} M$

$L > L_{Ed} \Rightarrow$ mass loss

L_{Ed} also appears in other areas of astrophysics, such as accreting n.s. (can have $L > L_{Ed}$ if radiation emerges outside accretion column).

For a high-mass ms star, $T_{eff} \sim 50{,}000\,K \Rightarrow$ H ionized in photosphere $\Rightarrow \bar{\kappa} = \kappa$ (electron scattering) $\Rightarrow L_{Ed} \simeq 1.5\times10^{31}\frac{M}{M_\odot}$ W

$\frac{L_{Ed}}{L_\odot} \simeq 3.8\times10^4 \frac{M}{M_\odot}$ Note: both grav. + rad. forces $\propto \frac{1}{r^2}$.

For $M = 90\, M_\odot$, find $L \sim \frac{1}{3}L_{Ed}$. Such large stars exhibit significant mass loss.

p. 342 Variations of Main-Sequence Stellar Parameters with Mass

Theoretical models of stars burning H in core fall right on observed m.s. (oh F 10.13 p. 342)

Note: the ms is not an evolutionary sequence. During its ms lifetime, the star moves only slightly (+ not along the ms).

Lower right end of ms is low-mass stars w/ $M \sim 0.072\, M_\odot$, $T \sim 1700\,K$, $L \sim 5\times10^{-4} L_\odot$

Upper left end: $M \sim 90 M_\odot$, $T \sim 53{,}000\,K$, $L \sim 1\times10^6 L_\odot$.

High-mass stars burn their fuel much faster, + leave the ms after only a million years as opposed to billions or trillions of years for

1a ~~high~~ low mass stars. (prob. 10.21).

High-mass stars are convective in the core + radiative outside, while low-mass stars are convective at the surface + radiative at the core, with the star being completely convective for $M < 0.3\, M_\odot$