THE COMPLEXITY OF HIGHER CHOW GROUPS

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Abstract. Let $X/\mathbb{C}$ be a smooth projective variety. We consider two integral invariants, one of which is the level of the Hodge cohomology algebra $H^*(X,\mathbb{C})$, and the other involving the complexity of the higher Chow groups $\text{CH}^*(X, m; \mathbb{Q})$, for $m \geq 0$. We conjecture that these two invariants are the same, and accordingly provide some strong evidence in support of this.

Introduction

Interest in the complexity of the higher Chow groups $\text{CH}^r(X, m; \mathbb{Q})$, particularly in the case $m = 0$, goes back to some seminal works of Mumford [11] and Griffiths [4]. Subsequent to this, were the ideas of Bloch, and later fortified by Beilinson, specifically when $m = 0$, that there should be a descending filtration, 

$$\text{CH}^r(X/\mathbb{C}, m; \mathbb{Q}) = F^0 \supseteq F^1 \supseteq \cdots,$$

whose graded pieces 

$$\text{Gr}^r F^\nu \text{CH}^r(X/k, m; \mathbb{Q}) \simeq \text{Ext}^\nu_{\text{MM}(k)(\text{Sp}(k), h^{2r-m-\nu}(X)(r))},$$

where Beilinson’s formulation involved the case $m = 0$, and where $h^*(r)$ is motivic cohomology. It is then reasonably clear, that the existence of such a filtration (called the (conjectural) Bloch-Beilinson filtration), is pivotal with regard to issues of complexity.

With this in mind, we begin with the following.

Definition 0.1. Let $X$ be a projective algebraic manifold, with Hodge cohomology $H^*(X, \mathbb{C}) = \oplus_{p,q} H^{p,q}(X)$. We define

$$\text{Level}(H^*(X, \mathbb{C})) = \text{Max} \{ p - q \mid H^{p,q}(X) \neq 0 \}.$$

$$\text{Level}(\text{CH}^*(X, m; \mathbb{Q})) = \text{Max} \{ \text{Level}(\text{CH}^r(X, m; \mathbb{Q})) \},$$

where $\text{Level}(\text{CH}^r(X, m; \mathbb{Q})) = 0$ for $r < m$, otherwise for $r \geq m$, 

$$\text{Level}(\text{CH}^r(X, m; \mathbb{Q})) = \min \{ \nu \geq 0 \mid \text{CH}^r(X, m; \mathbb{Q}) \rightarrow \text{CH}^r(X \setminus Y, m; \mathbb{Q}) \text{ is zero} \},$$

$Y \hookrightarrow X$ closed, $\text{codim}_X Y = r - \nu - m$.

The expected relationship between these invariants is:

Conjecture 0.2. For all $m \geq 0$,

$$\text{Level}(H^*(X, \mathbb{C})) = \text{Level}(\text{CH}^*(X, m; \mathbb{Q})).$$


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Based on conjectural assumptions, an outline of a proof Conjecture 0.2 in the case where \( m = 0 \) appeared in [5](Cor. 15.64). In this paper, we provide the full details for all \( m \geq 0 \); the case \( m > 0 \) requiring some new ingredients, which should be of interest to the reader. Finally, we exhibit a class of examples involving complete intersections, based on new ideas from [10].

Our main result is:

**Theorem 0.3.** Let \( X/\mathbb{C} \) be smooth and projective. Further, assume that:

- 1. The GHC holds.
- 2. For any smooth projective variety \( Y \) defined over \( \mathbb{Q} \), the Abel-Jacobi map
  \[ \Phi_{r,m} : \text{CH}_{\text{hom}}^r(Y/\mathbb{Q}, m; \mathbb{Q}) \to \text{Ext}_{\text{MHS}}^1(\mathbb{Q}(0), H^{2r-m-1}(Y(\mathbb{C}), \mathbb{Q}(r))). \]
- 3. Either \( m \leq 2 \), or for a given \( m \geq 3 \), there exists a projective algebraic manifold \( B \) of dimension \( m-1 \), and a class \( \gamma \in H^{m-1}(B, \mathbb{R}(m-1)) \), with \( \gamma^{m-1,0} \neq 0 \), in the image of the regulator map \( r_D : \text{CH}^m(B, \mathbb{Q}(m)) \otimes \mathbb{R} \to H^m_D(B, \mathbb{R}(m)) \simeq H^{m-1}(B, \mathbb{R}(m-1)). \)

Then:

\[ \text{Level}(H^*(X/\mathbb{C})) = \text{Level}(\text{CH}^*(X, m; \mathbb{Q})). \]

1. **Notation**

(i) \( \text{CH}^r(X, m) = H^r_{\text{MHS}}(X, m; \mathbb{Q}) \) are the higher Chow groups introduced in [2]. We put \( \text{CH}^r(X, m; \mathbb{R}) = \text{CH}^r(X, m) \otimes \mathbb{R} \).

(ii) A full explanation of the Hodge conjecture, the general Hodge conjecture (GHC), and the hard Lefschetz conjecture appear in [8](Ch. 7, 15).

(iii) Real Deligne (co-)homology \( H^*_D(X, \mathbb{R}(m)) \) is explained in [9].

(iv) Let \( \mathbb{A} \subseteq \mathbb{R} \) be a subring. The Tate twist is given by \( \mathbb{A}(m) = (2\pi)^m \cdot \mathbb{A} \).

(v) The reader is assumed to be familiar with mixed Hodge structures (MHS).

(vi) \( \{NP^iH^i(X, \mathbb{Q})\}_{p \geq 0} \) is the coniveau filtration as defined in [8](Ch. 7).

(vii) Given a family of varieties \( \{X_t\}_{t \in S} \), where \( S \) is a base variety, a general member of this family refers to an \( X_t \), \( t \in U \), where \( U \subset S \) is a non-empty Zariski open subset, determined by certain “generic” properties, such as \( X_t \) nonsingular, etc.

2. **Proof of Theorem 0.3**

We introduce our first key ingredient.

**Theorem 2.1** ([9]). Let \( X \) be a projective algebraic manifold. Assume the following:

(i) The hard Lefschetz conjecture holds.

(ii) Either \( m \leq 2 \), or for a given \( m \geq 3 \), there exists a projective algebraic manifold \( B \) of dimension \( m-1 \), and a class \( \gamma \in H^{m-1}(B, \mathbb{R}(m-1)) \), with \( \gamma^{m-1,0} \neq 0 \), in the image of the regulator map \( r_D : \text{CH}^m(B, \mathbb{Q}(m)) \otimes \mathbb{R} \to H^m_D(B, \mathbb{R}(m)) \simeq H^{m-1}(B, \mathbb{R}(m-1)). \)

Then:

\[ \text{Level}(H^*(X/\mathbb{C})) = \text{Level}(\text{CH}^*(X, m; \mathbb{Q})). \]

**Corollary 2.2.** Let us assume the GHC and Theorem 2.1(ii). Then:

\[ \text{Level}(H^*(X/\mathbb{C})) \leq \text{Level}(\text{CH}^*(X, m; \mathbb{Q})). \]

**Proof.** Left to the reader. \( \square \)

Next, let \( V/\mathbb{Q} \) be a smooth quasi-projective variety defined over \( \mathbb{Q} \). Now based on a Bloch-Beilinson conjecture assumption in the case \( m = 0 \), it is conjectured in [6] that:
One can show Remark 2.4.

The Abel-Jacobi map Conjecture 2.3.

Let us assume the GHC and Conjecture 2.3. Then for any integer

\[ \{ \overline{X} \} \]

our assumptions, there is, for any smooth projective variety

Proof. Using primarily [1], together with the works of [7], [5], and [6], and together with

According to Beilinson, such a decomposition lifts to an idempotent decomposition

\[ \Delta_X = \bigoplus_{p+q=2n} \Delta_X(p, q) \in \text{CH}^n(X \times X; \mathbb{Q}), \]

called a Chow-K"unneth decomposition in the sense of [12]. Now let \( \ell = \text{Level}(H^*(X, \mathbb{C})) \). One can easily argue that \( \ell \leq n \). Note that

\[ \text{CH}^\ell(X, m; \mathbb{Q}) = \sum_{p+q=2n} \Delta_X(p, q) \cdot \text{CH}^\ell(X, m; \mathbb{Q}). \]

It will suffice (Proposition 2.6 below) to determine the level of \( \Delta_X(p, q) \cdot \text{Gr}^{\nu} \text{CH}^\ell(X, m; \mathbb{Q}) \).
Case $p \leq \ell$. Then $q = 2n - p \geq n$. Thus $H^q(X, \mathbb{Q}) = L^{n-q}_X H^p(X, \mathbb{Q})$, hence we may assume that $|\Delta_X(p, q)| \subset X \times V$, where if we want, by Bertini, $V$ is smooth and $\text{codim}_X V = n - p$. It follows that
\[
\text{Level}(\Delta_X(p, q); \text{CH}^r(X, m; \mathbb{Q})) \leq (r - n - m) + p \leq p \leq \ell.
\]

Case $\ell < p \leq n$. Thus $p = \ell + k$, where $k \geq 1$. Note that $\text{Level}(H^p(X, \mathbb{Q})) \leq \ell$. For the sake of simplicity, let us assume that $\text{Level}(H^p(X, \mathbb{Q})) = \ell$; the general case being similar. Then as one would expect, from Hodge theory, $\text{GHC}$. Thus we can assume that $\oplus \nabla_X Y$ to the case where $r > n$. Hence, $r > q$. Finally, if we now assume that $k > 0$, where as in the previous case, we may assume that $k > n$ is the following. We have the K"unneth decomposition
\[
\Delta = \bigoplus_{p+q=2n} [\Delta(p, q)],
\]
where $\Delta(p, q)$ satisfies the condition of the above theorem. Thus
\[
\Delta' := \bigoplus_{p+q=2n} \Delta(p, q) \sim_{\text{hom}} \Delta.
\]
Thus we are reduced to the following:

**Proposition 2.6.** Let $\Xi \in \text{CH}^{2n}_{\text{hom}}(X \times X; \mathbb{Q})$. Then
\[
\text{Level}(\Xi; \text{CH}^r(X, m; \mathbb{Q})) \leq \ell.
\]

**Proof.** Let $F^r := F^r \text{CH}^r(X, m; \mathbb{Q})$. Note that $F^r = \text{Gr}^r \text{CH}^r(X, m; \mathbb{Q})$. Since $\text{CH}^{2n}_{\text{hom}}(X \times X; \mathbb{Q}) = F^1 \text{CH}^1(X \times X; \mathbb{Q})$, then $\Xi, F^r = 0$. Consider the short exact sequence
\[
0 \rightarrow F^r \rightarrow F^{r-1} \rightarrow \text{Gr}^{r-1} F^r \rightarrow 0.
\]
Then $\Xi, F^{r-1} \rightarrow 0 \in \text{Gr}^{r-1} F^r$, hence $\Xi, F^{r-1} \subset F^r$, where we know that $\text{level}(F^r) \leq \ell$. Proceeding by downward induction, consider the short exact sequence
\[
0 \rightarrow F^r \rightarrow F^{r-1} \rightarrow \text{Gr}^{r-1} F^r \rightarrow 0.
\]
By induction $\text{Level}(F^r) \leq \ell$, and $\Xi, F^{r-1} \rightarrow 0 \in \text{Gr}^{r-1} F^r$. Thus $\text{Level}(\Xi, F^{r-1}) \leq \ell$. \qed
3. An explicit example

Let $X/C \subseteq \mathbb{P}^{n+r}$ be a general complete intersection of multidegree $(d_1, \ldots, d_r)$ and $\Omega_X(k)$ be its (nonsingular!) Fano variety of $k$-planes, and suppose $X = Z \cap \mathbb{P}^{n+r}$, where $Z \subseteq \mathbb{P}^{n+r+1}$ is a general complete intersection of multidegree $(d_1, \ldots, d_r)$. The geometric properties of such complete intersections are explored elsewhere (e.g., [3], [10], [8]). Set $\delta := (k+1)(n+r-k) - \sum_{j=1}^{r} \binom{d_j+k}{k}$, $l = k(n+1+r-k) + r - \sum_{j=1}^{r} \binom{d_j+k}{k}$ and assume $\delta \geq n - 2k \geq 0$. Observe that $\delta = n - 2k + l$. Indeed $\delta = \dim \Omega_X(k)$, and through a generic point of $Z$ passes an $l$-dimensional family of $\mathbb{P}^k$’s. Let $\Omega_Z$ be the subvariety of $\Omega_Z(k)$ obtained by the intersection of $l$ general hyperplane sections. By Bertini’s theorem, and a dimension count, $\Omega_Z$ is smooth and irreducible of dimension $n + 1 - k$. If we set $\Omega_X = \Omega_Z \cap \Omega_X(k)$, then $\Omega_X$ is smooth of pure dimension $n - 2k$. Consider the following diagram:

![Diagram](image)

where $P(X), P(Z)$ are $\mathbb{P}^k$-bundles, $\tilde{X} = \pi_Z^{-1}(X)$ and all the maps depicted are natural projections, except $i_0, i, j$, which are inclusions.

**Proposition 3.1 ([3]).** Assume $\delta \geq n - 2k \geq 0$. Then the cylinder homomorphism

$$\Phi_* := \pi_{X,*} \circ \rho_X^* : H^{n-2k}(\Omega_X, \mathbb{Q}) \to H^n(X, \mathbb{Q}),$$

is surjective.

Set $Y = \pi_X(P(X))$. Then $Y \subseteq X$ is a subvariety of codimension $k$. We conclude:

**Corollary 3.2.** The natural map $H_n(Y, \mathbb{Q}) \to H_n(X, \mathbb{Q})$ is surjective. Hence, cycles on $X$ are supported on a subvariety of codimension $k$.

The corollary above can be restated as:

**Corollary 3.3.** The general Hodge conjecture $GHC(k, n)$ is true for $X$.

Next, let us write $X = V(F_1, \ldots, F_r) \subseteq \mathbb{P}^{n+r}$, for generic choice of $\{F_1, \ldots, F_r\}$, and $W = V(F_1, \ldots, F_{r-1}) \subseteq \mathbb{P}^{n+r}$, with inclusion $j : X \hookrightarrow W$.

**Theorem 3.4. ([10])** Assume that $\delta \geq n - 2k \geq 0$. Then the cylinder map

$$\Phi_{X,*} : \text{CH}^{r-k}(\Omega_X, m; \mathbb{Q}) \to \text{CH}^r(X, m; \mathbb{Q})/j^* \text{CH}^r(W, m; \mathbb{Q}),$$

is surjective.

Now observe that $r > n + m - k \Rightarrow r - k > n - 2k + m > \dim \Omega_X + m$, hence $\text{CH}^{r-k}(\Omega_X, m; \mathbb{Q}) = 0$. Thus we will assume that $r \leq n + m - k$. Now choose $k$ such that $\text{Level}(H^*(X, \mathbb{Q})) = n - 2k$, and assume that $\delta \geq n - 2k$. Further, put $\nu := \text{Level}(\text{CH}^r(X, m; \mathbb{Q})/j^* \text{CH}^r(W, m; \mathbb{Q}))$. 


By definition of $\nu$, observe that $\nu \leq r - \text{codim}_X Y - m = r - k - m$. But $r \leq n + m - k$, hence $\nu \leq n - 2k$. Finally, $W$ is of a lower order that $X$, and one can argue via an inductive argument, that $\text{Level}(j^*\text{CH}^r(W,m;\mathbb{Q})) \leq n - 2k$. From Corollaries 2.2 and 3.3 we deduce:

**Corollary 3.5.** If $X$ is a general complete intersection satisfying $\delta \geq n - 2k \geq 0$, where $k \geq 0$ is given such that $\text{Level}(H^*(X,\mathbb{Q})) = n - 2k$, then

$$\text{Level}(H^*(X,\mathbb{C})) \geq \text{Level}(\text{CH}^r(X,m;\mathbb{Q})),$$

and we have an equality in the case $m < 3$.

**References**