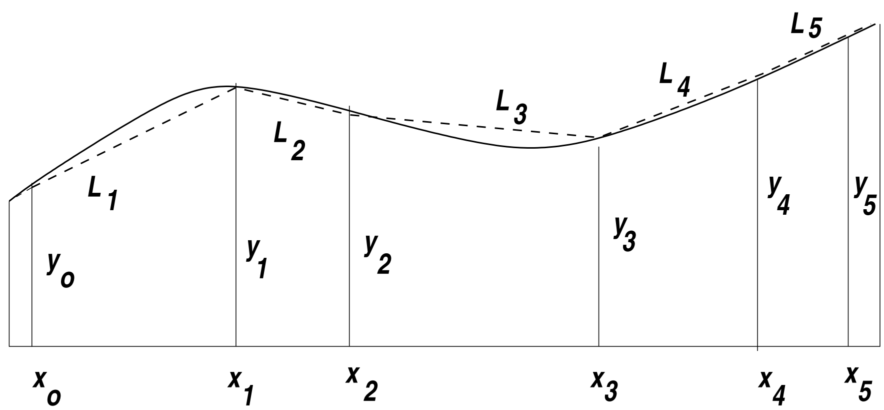
Arc-Length

P. Coulton *

The arc-length of the curve of a function can be estimated by choosing a partition of the interval $[a, b]$, say $\{x_o, x_1, \ldots, x_n\}$ and computing the length of the secant line for each subinterval. As an example consider the case given by



$$L_1 = \sqrt{(x_1 - x_o)^2 + (y_1 - y_o)^2} \text{ and } L \approx L_1 + L_2 + L_3 + L_4 + L_5.$$

In general we can write

$$L \approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}.$$

This simpiifies to

$$L \approx \sum_{i=1}^{n} \sqrt{1 + (\frac{\Delta f}{\Delta x})^2} \quad \Delta x,$$

where we have assumed that the subintervals are all of length $\Delta x = (b-a)/n$. Note that $\Delta f / \Delta x$ is the slope of the secant line and thus we may use the

*Author, E-mail: prcoulton@eiu.edu

mean value theorem to pass to the integral. So we define the arc-lentgh as

$$L = \int_a^b \sqrt{1 + f'(x)^2}\; dx$$

Let $f(x) = \sqrt{1 - x^2}$, ie.e., the semi-circle. Then

$$\sqrt{1 + f'(x)^2} = \sqrt{1 + \frac{x^2}{1 - x^2}} = \frac{1}{\sqrt{1 - x^2}}.$$

The arc length integral is

$$L = \int_{-1}^{1} \frac{1}{\sqrt{1 - x^2}}\; dx = \arcsin(1) - \arcsin(-1) = \frac{\pi}{2} - \frac{-\pi}{2} = \pi,$$

which is the usual value for the arc-length of the semi-circle of radius 1.

Department of Mathematics
Eastern Illinois University