Constructing Axially Invisible Objects

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Abstract

We investigate the properties of mirrored objects in space that are invisible for some fixed direction. By invisible in a fixed direction we mean that light rays parallel to the axis of the object will bend around the object and exit the object along the same straight line that they first stuck the object. We call these objects *axially invisible objects*. The advantage of such directionally invisible objects is that they would be difficult to detect and they would be resistant to light ray pressure in the specified axial direction.

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1. Introduction

We say that an object in $\mathbf{R}^n$ is *axially invisible* in a given axial direction (say, the y-axis direction) if every ray moving in that direction can be diverted around the object so that the ray exits the object in the same direction and along the same line that it entered the object except possibly on a set of measure zero.

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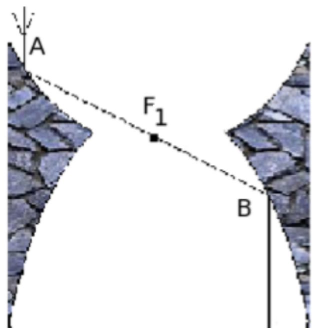

Figure 1. *An example of a parabolic lens system.*

We say that an object constructed from opposing parabolic curves is a *parabolic lens system* with respect to the y-axis if the opposing parabolic curves have a common focus at $(0, t)$ and the set of all rays parallel to the y-axis that strike the upper parabolic curve pass through the focus and strike the lower parabolic curve so that the resultant outgoing ray is also parallel to the y-axis.

Proofs in the article are devoted primarily to the planar case; however, these proofs are easily adapted to higher dimensional cases. We begin with the following lemma. The proof is left to the reader.

Lemma 1. *The equation of a concave up parabola with focus at* $(0, p+r)$ *is given by*

$$4py = x^2 + 4pr, \text{ or } \; y = \frac{x^2}{4p} + r,$$

and the equation of a concave down parabola with focus at $(0, p+r)$ *is given by*

$$4py = -x^2 + 4p(2p+r), \text{ or } \; y = -\frac{x^2}{4p} + 2p + r.$$

2. Parabolic System Objects PS^2.

We define a *mirrored parabolic system* $PS^2(a, b)$ in the plane as the object determined by a set of parabolic mirrors on the surfaces of the given regions:

$$A_+ = \{(x, y) \mid x \in [-a, -2b] \cup [2b, a], \; -\frac{x^2}{4b} + b \leq y \leq \frac{x^2}{4b} - b\}$$

and

$$A_- = \{(x, y) \mid x \in [-a, -2b] \cup [2b, a], -\frac{x^2}{4b} + 3b - \frac{a^2}{2b} \leq y \leq \frac{x^2}{4b} + b - \frac{a^2}{2b}\}.$$

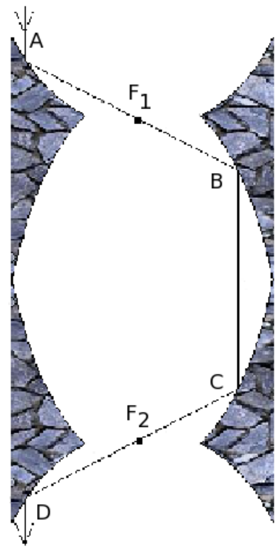

Figure 2. *Parabolic lens in series giving* $PS^2(a, b)$*.*

Proposition 2. *The object* $PS^2(a, b)$ *is axially invisible with respect to the* y*-axis direction. In particular, if a ray from infinity parallel to the* y*-axis strikes the object* $PS^2(a, b)$ *on the parabolic curves at a point* $A = (x_o, y_o)$*, then the ray is reflected so that the ray exits parallel to the* y*-axis at a point* $D = (x_o, -y_o + 2b - \frac{a^2}{2b})$*.*

Proof: Observe that the focus of the upper system is at $F_1 = (0, 0)$ and the focus of the lower system is at $F_2 = (0, 2b - a^2/(2b))$. We say that the object is in *series* when one lens system is connected to the other lens system in a consecutive fashion such that first lens system ends at the two points $(-a, b - a^2/(4b^2))$ and $(a, b - a^2/(4b))$ and the second lens system begins at these points as shown in Figure 2.

Since the parabolic curve acts as a mirror, we know that a ray parallel to the y-axis which strikes the upper parabolic curve $x^2/4b - b$ at the point $A = (x_o, y_o)$, will pass through the focus of the parabola at F_1. The ray will then strike the point $B = (-x_o, -y_o)$ which is on the lower parabolic curve $y = -x^2/4b + b$.

Since the ray has passed through the focus, F_1, of this parabola at $(0, 0)$, it follows that the ray will be parallel to the y-axis after reflection. Now if the two objects are placed in series, that is, the ray passes first through the A_+ lens and then through the A_- lens, it will necessarily follow that the ray will pass through the second focus at $F_2 = (0, 2b - a^2/2b)$. This ray will strike the lower parabolic mirror at a distance y_o below the lower focus. In other words, the ray will strike the point $D = (x_o, -y_o + 2b - a^2/2b)$. By symmetry, the ray will exit parallel to the y-axis in the same line as the initial ray. This completes the proof. ▷

Corollary 3. *Consider the object $PS^2(a,b)$ rotated about its axis of symmetry obtaining a solid of revolution denoted by $PS^3(a,b)$. Then $PS^3(a,b)$ is invisible with respect to the y-axis direction.*

Proof: Since the new object $PS^3(a,b)$ is the volume of revolution of $PS^2(a,b)$ the same tracing arguments hold as in the proof of Proposition 2.This completes the proof. ▷

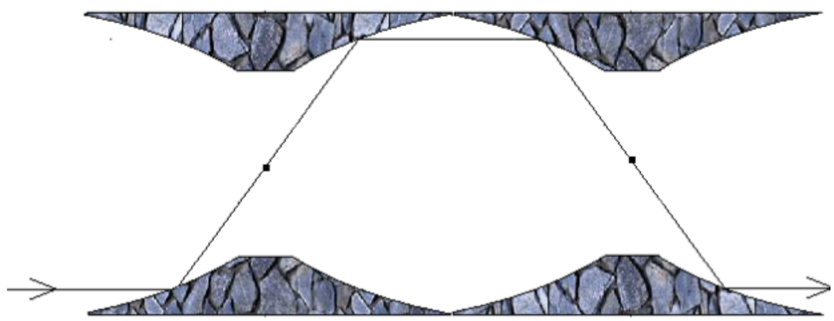

Figure 3. *Two cutaway parabolic mirrors in series.*

Consider the regions defined by

$$B_+ = \{(x,y) \mid x \in [-a,-c] \cup [c,a],\ -\frac{x^2}{2b} + b \le y \le \frac{x^2}{2b} - b\},$$

and B_- a copy of the same object translated downward in the y-axis direction by $2b - a^2/2b$. We will refer to this object as $PS^2(a,b,c)$.

Proposition 4. *The cutaway parabolic object $PS^2(a,b,c)$ is invisible with respect to the y-axis direction and the volume of revolution $PS^3(a,b,c)$ obtained by rotating the object around the axis of symmetry (say, the y-axis) is invisible with respect to the y-axis direction.*

Proof: Observe that if $c = 2b$, then the object is the same as the one given in Proposition 2. The same argument holds as given in Proposition 2 except that the choice for (x_o, y_o) is restricted to to smaller set. The same tracing arguments work as in Proposition 2 for the given points $x_o, y_o)$ on the parabolic surfaces. This completes the proof. ▷

We observe that the three-dimensional objects obtained by the volumes of revolution for $PS^2(a,b)$ and $PS^2(a,b,c)$ when rotated about the $x = a$ axis (equivalently, the $x = -a$ axis) are also axially invisible with respect to the y-axis direction.

3. Combination Parabolic System Objects

Here we turn our attention to components of invisible objects which are not necessarily symmetric with respect to reflection through the x-axis. Consider the lens system LS^2 defined by the region in the plane determined by

$$C = \{(x,y) \mid x \in [-c,-r] \cup [r,c],\ -\frac{x^2}{4b} + b \leq y \leq \frac{x^2}{4a} - a\},$$

where $r = 2\sqrt{ab} < c$ and $a \leq b$. Note that the two parabolas have the same focus but different focal lengths when $a < b$. The distance from the vertex of the parabola to the focus for the upper lens is a an the distance from the vertex to the focus of the lower lens is b.

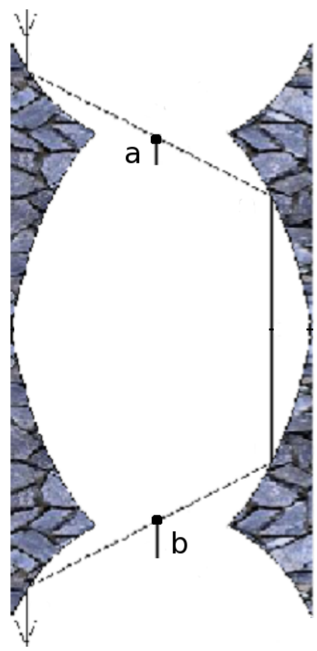

Figure 4. A pair of lenses in series.

Proposition 5 *If a ray parallel to the y-axis strikes the upper mirror of LS^2 at a point (x_o, y_o), then the ray passes through the focus at $(0,0)$ and strikes* *the lower mirror at a point* (x_1, y_1) *satisfying*

$$y_1 = -\frac{2by_o^2}{x_o^2}\left(1+\sqrt{1+x_o^2/y_o^2}\,\right), \text{ and } \ x_1 = \frac{x_o y_1}{y_o},$$

such that the exiting ray is parallel to the y*-axis.*

Proof: It is straight forward to check that the focus is at $(0,0)$ for both parabolic curves. Next, observe that the reflected ray moves along the line given by

$$x = \frac{x_o}{y_o}y = \frac{4ax_o y}{x_o^2 - 4a^2}, \text{ where } y_o = \frac{x_o^2 - 4a^2}{4a}.$$

Substituting into the equation for the lower parabolic curve gives

$$y = -\frac{x_o^2}{4y_o^2 b}\, y^2 + b.$$

Simplifying this relation gives the equation for y_1 under the observation that the value of $y_1 < 0$. Using the equation of the line gives

$$x_1 = -\frac{2by_o}{x_o}\left(1+\sqrt{1+x_o^2/y_o^2}\,\right)$$

Finally we wish to show that every ray parallel to the y-axis that strikes the top parabolic curve will reflect through the focus and strike at some point of the bottom parabolic mirror. To accomplish this, we must show that the value of y when $x = c$ for the bottom parabolic curve is less than the absolute value of y when $x = c$ is on the top parabolic curve.

If this is the case, then a straight line from the lowest point on the lower parabola through the focus will strike some point on the upper parabola. We let y^+ denote the top value of the upper parabola and y^- denote the absolute value of y for the lowest point of the lower parabola, then

$$y^+ = \frac{c^2 - 4a^2}{4a}, \text{ and } \ y^- = \frac{c^2 - 4b^2}{4b}.$$

We wish to show that $y^+ > y^-$ when $a < b$ and $2\sqrt{ab} < 2b \leq c$. Let $\tau > \sqrt{b/a}$ and substitute $c = 2\tau\sqrt{ab}$, then

$$y^+ = \tau^2 b - a, \text{ and } \ y^- = \tau^2 a - b.$$

Observe that $y^+ > y^-$ when $\tau = \sqrt{b/a}$ and differentiation with respect to τ shows that y^+ grows faster than y^-. This completes the result. $\triangleright$

If one place the parabolic mirror with the larger focal length on the top, then the of the bottom lens is wider than the width of the top of the lens. On the other hand, it is possible to construct a new lens with the curves used in Proposition 2 on either side of the original lens. In this way an axially invisible object can be constructed using the lens from Proposition 5.

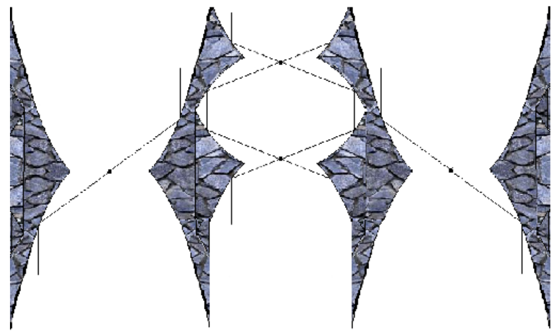

Figure 4. An example of an odd symmetry lens embedded in an axially invisible object.

It is apparent that from this type of lens system it is possible to construct three dimensional lens system objects using rotation about an axis of symmetry.

4. Elliptic Combination Lenses.

We consider the two mirrored curves

$$y = \frac{b}{2a^2}x^2 - \frac{a^2}{2b}, \quad \text{with } 0 \leq y \leq p.$$

and

$$y = -\sqrt{b^2 - a^2} \pm \frac{b}{a}\sqrt{a^2 - x^2} \quad \text{with } -2\sqrt{b^2 - a^2} \leq y \leq 0,$$

Proposition 6 *Assume that an object is constructed from the parabolic and elliptic curves given above. A ray parallel to the y-axis direction that strikes the upper parabolic curve passes through the focus at $(0, 0)$ and strikes the lower elliptic surface at some point (x_1, y_1) with $y_1 < 0$ and after reflecting passes through the lower focus of the ellipse at $y = -2\sqrt{b^2 - a^2}$.*

Proof: We need only show that the two curves have a common focus at $(0, 0)$ and that the two curves meet at $(\pm a^2/2b, 0)$. Observe that the focus of the parabola is a distance $2a^2/b$ above the vertex which is at $(0, a^2/2b)$. Therefore, the focus of the parabolic curve is at $(0, 0)$.

The focus of the elliptic curve is at a distance of $\sqrt{b^2 - a^2}$ above the midpoint of the ellipse which is given by the defining equation at the point $(0, -\sqrt{b^2 - a^2})$. We conclude that the two conic sections have a common focus at $(0, 0)$. When $y = 0$ in the parabolic equation, we obtain $x = \pm a^2/b$. Now oberve that replacing $x = \pm a^2/b$ in the elliptic equation gives $y = 0$. We conclude that the two conic sections meet at the desired points. This completes the proof. $\triangleright$

Proposition 7 *Consider a lens combination constructed from two identical combinations lenses as defined in Proposition 6 with the two elliptic curves in series and the two parabolic curves at the ends such that the width of the parabolic curves is greater than or equal to $2a$. Then a pair of these combination lenses is axially invisible with respect to the y-axis direction.*

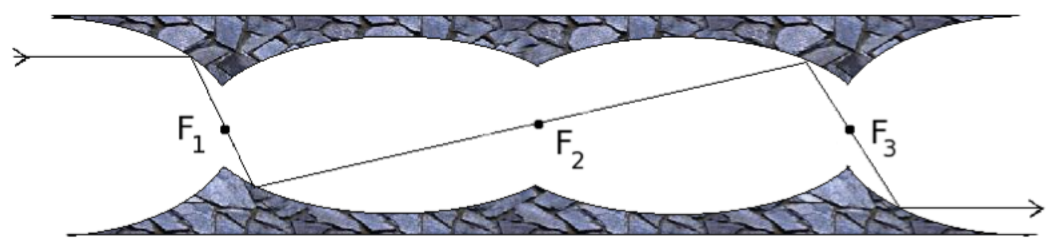

Figure 5. A combination lens is shown on its side with the elliptic curves in series in the interior.

Proof: A ray parallel to the y-axis hits the parabolic curve at some point (x_o, y_o). The ray then passes through the foci at $(0, 0)$ and at $(0, -2\sqrt{b^2 - a^2})$. The ray then hits the second parabolic mirror at $(-x_o, -2\sqrt{b^2 - a^2} - y_o)$ and is parallel to the y-axis. If a second lens is placed in series with this lens, then the ray will be reflected until it hits a point on the last parabolic curve with x coordinate x_o. The ray will then be reflected and return to the line of its original trajectory. Therfeore the combination of lenses is axially invisible with respect to the y-axis direction, which is what we wished to prove. $\triangleright$