Chaotic Dynamics of Base-n Expansions

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Abstract

One of the more interesting properties of chaotic dynamics is the way in which the system appears to mimic a stochastic process. A stochastic process is by nature an indeterminate system. On the other hand, a properly designed chaotic system can produce the same apparent outcomes as a stochastic system. This may induce to the untrained eye the very same effect as one would expect in stochastic system, when in fact, these chaotic dynamical systems are strictly deterministic.

In this paper, we consider a simple model of a chaotic discrete dynamical system and show that the chaotic dynamics is a result of the fact that the initial conditions for the system are in fact a parameter space for a stochastic system sample space.

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1. Introduction

Recall that a stochastic process is an experiment which can be performed repeatedly without end and which has the same set of outcomes for each experiment with a fixed probability distribution for the outcomes of each experiment. We will assume that the outcomes of each experiment are finite and that the probability distribution is uniform.

A simple example of this process is given by a game spinner with three regions labeled A, B, and C. When spun the pointer is assumed to land in

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each of the regions with an equal probability of 1/3. If $S = \{A, B, C\}$, then the sample space for a countably infinite set of trials is

$$U = \{< X_1, X_2, X_3, \ldots > \quad | \quad X_i \in S, \ \forall \ i \in \mathbf{Z}^+\}.$$

Now consider any continuous positive surjective function $f : [0,1] \to [0,1]$ and define the iterative sequence

$$x_1 = x_1, \quad x_{i+1} = f(x_i).$$

The sequence generated by the starting point, or seed value x_1, is illustrated in Figure 1.

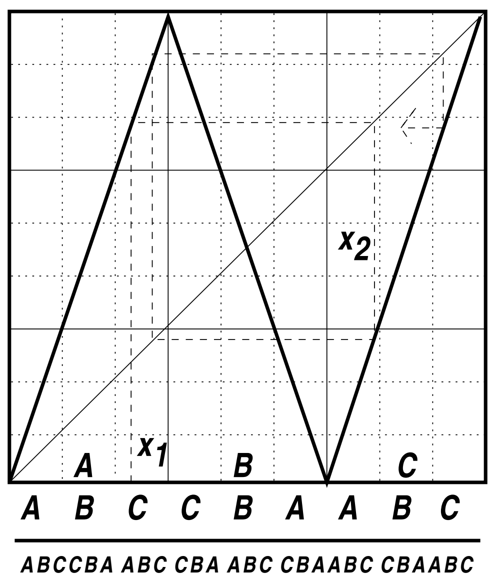

Figure 1. *Tracing the iteration method.*

As an example, consider the surjective function $f : [0,1] \to [0,1]$ as in Figure 1 defined by:

$$f_3(x) = \begin{cases} 3x & : \ x \in [0, 1/3] \\ -3x + 2 & : \ x \in (1/3, 2/3) \\ 3x - 2 & : \ x \in [2/3, 1] \end{cases}$$

We have split the domain in Figure 2 into three basic regions in order to study the sequence of iterations of the function. The reduction is as follows:

- if $x_i \in [0, 1/3]$, then $X_i = A$;
- if $x_i \in (1/3, 2/3)$, then $X_i = B$;
- if $x_i \in [2/3, 1]$, then $X_i = C$.

We call x_1 the *seed* of the sequence. We observe that for each $x_1 \in [0, 1]$ the iteratively defined sequence is an element of U.

Now observe that if x_1 is in $[0, 1/9]$ then x_1 corresponds to A and x_2 corresponds to A. on the other hand, if x_1 is in the interval $(1/9, 2/9)$ then x_1 corresponds to A and x_2 corresponds to B. If x_1 is in the interval $[2/9, 1/3]$, then x_1 corresponds to A and x_2 corresponds to C, and so on.

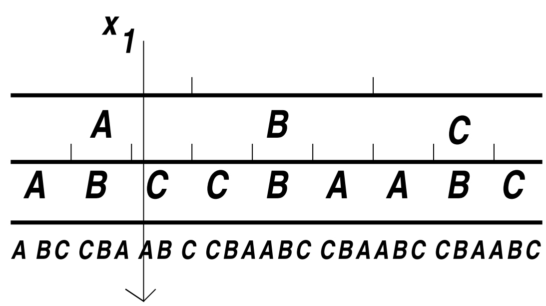

Figure 2. *An example of Lemma 1 for* $x_1 \to < A, C, A, ... >$.

An example of the first few steps in this dynamical system process is given in Figures 1 and 2. The dotted line in Figure 1 shows the iteration process for a specific seed value. Once x_1 is chosen, then the sequence is determined and can be obtained from Figure 2 by drawing a vertical line through the chart at x_1 and reading off the corresponding interval symbols in the chart. We will refer to this sequence as the symbol sequence.

Procedure 1. *The symbol sequence for the seed value* $x_1 \in [0, 1]$ *for the surjective function* $f_3(x)$ *is given by the symbols read in order for the vertical line through* x_1 *as shown in the chart of Figure 2 at* x_1*.*

Concept: From the form of the function $f_3(x)$ it is clear that it is possible to go from any state, say A, to any other state in the next step. For a point chosen at random, the probability of a transition from a given state to any other state is exactly 1/3. From the arguments above, we will assume that the symbol sequence outlined, in the first n steps of the iteration, corresponds to the chart given.

Now suppose that in the n-th step we are in the interval corresponding to the state A. There is a transition from this state to the next state with probability $1/3$ for each outcome. This means that we can divide the current interval into three parts. According to the rule of the chart, the transition from an interval labeled A to the next interval symbol will be either in the form $A|B|C$ or $C|B|A$. Now, observe that each time we visit a region labeled B the order of the interval division, i.e., the form $A|B|C$ or $C|B|A$, is reversed.

The rule for the choice of label then is determined by the number of times the symbol B has appeared in the sequence up to the n-th step. If the number is even then the symbols in the interval division division to the next step looks like $A|B|C$ and if the number is odd, then it will appear as $C|B|A$. This corresponds to the transitions that occur in the function iteration. This is precisely the rule in the self replicating form of the chart in Figure 2. Therefore the label regions that the vertical line passes through will correspond to the regions visited in proper order for the symbol sequence starting at x_1.

Proposition 2. *The symbol sequence at $x_1 \in [0, 1]$ gives a method for obtaining the ternary expansion of the decimal numbers $x \in [0, 1]$.*

Proof: Given a symbol sequence $< X_1, X_2, X_3, \ldots >$ constructed from the symbols A, B and C using the seed x_1 in $f_3(x)$, we construct the ternary, or base 3, expansion using the following rules:

- If $X_n = B$ then take $a_n = 1$;
- If $X_n = A$ and there are an even number of B's in the sequence up to X_{n-1}, then take $a_n = 0$ else take $a_n = 2$;
- If $X_n = C$ and there are an even number of B's in the sequence up to X_{n-1}, then take $a_n = 2$ else take $a_n = 0$.

We claim that the value of $x_1 \in [0, 1]$ satisfies

$$x_1 = \sum \frac{a_n}{3^n}.$$

It is only necessary to observe that the chart for the dynamical system corresponds to the Cantor set with respect to the interval associated to B in each case. The only difference being that in the ternary expansion there is never a reversal of the digits 0|1|2 in the interval decomposition. This anomaly is rectified by the rule for computing the ternary expansion by counting the number of reverses in the symbol sequence.

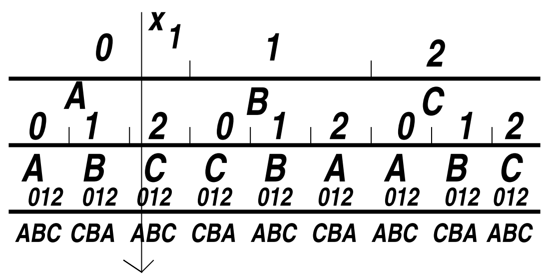

Figure 3. *The symbol sequence compared to the ternary expansion.*

Proposition 3. *The symbol sequence resulting from the function $f_3(x)$ has a period three cycle, which implies that it is chaotic.*

Proof: To see that the system is chaotic, recall that period three implies chaos[2, p. 986] and observe the following period three cycle:

$$f_3(\frac{2}{7}) = \frac{6}{7}; \quad f_3(\frac{6}{7}) = \frac{4}{7}; \quad f_3(\frac{4}{7}) = \frac{2}{7}.$$

The symbol sequence associated to $x_1 = 2/7$ is periodic and of the form

$$< A, C, B, A, C, B, \dots >$$

which is a period three cycle. This completes the proof.

We let

$$x_1 \simeq < A, C, B, A, C, B, \dots >$$

denote the relation between the seed x_1 and its symbol sequence. Now consider the dynamical systems given by function iterations obtained from $f_k : [0, 1] \to [0, 1]$ defined by

$$f_n(x) = \begin{cases} nx & : \text{ for } x \in [0, 1/n] \\ -n(x - 2/n) & : \text{ for } x \in (1/n, 2/n] \\ n(x - 2/n) & : \text{ for } x \in (2/n, 3/n] \\ \dots & : \dots \\ -n(x - 1) & : \text{ for } x \in ((n-1)/n, 1] \end{cases}$$

for n even and

$$f_n(x) = \begin{cases} nx & : \text{ for } x \in [0, 1/n] \\ -n(x - 2/n) & : \text{ for } x \in (1/n, 2/n] \\ n(x - 2/n) & : \text{ for } x \in (2/n, 3/n] \\ \dots & : \dots \\ n(x - (n-1)/n) & : \text{ for } x \in ((n-1)/n, 1] \end{cases}$$

for n odd.

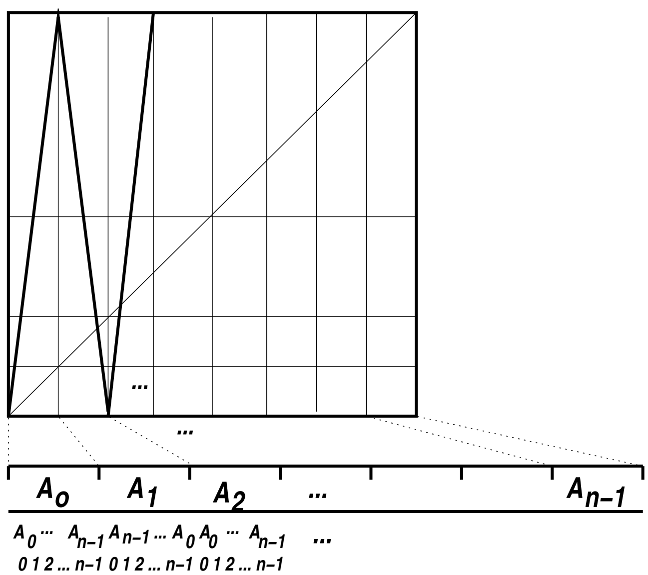

Fig. 5. *The base n dynamical system on the unit interval.*

For the binary, i.e. base 2, system a period three cycle is given by the seed value $x = 2/9$. To see this, observe that

$$f_2(2/9) = 4/9; \qquad f_2(4/9) = 8/9; \qquad f_2(8/9) = 2/9.$$

This implies that $2/9$ has the symbolic sequence

$$\frac{2}{9} \simeq < A, A, B, A, A, B, \dots > .$$

If we apply the following rule

1. Replace A by 0 if an even number of B's precede it otherwise replace the A with 1;

2. Replace B by 1 if an even number of B's precede it otherwise replace the B with 0;

then the expansion $< A, A, B, A, A, B, \ldots >$ gives the binary representation of 2/9, namely

$$(0.00111000111000111\ldots).$$

We note that this dynamical system is related to the Chebyshev-von Neuman-Ulam map

$$f_C(x) = 4x(1-x).$$

It should now be evident that if $f(x)$ is a mapping from $[0,1]$ with n linear switchbacks functions such that each piecewise linear monotonic function is onto $[0,1]$, then dynamical system is related to the base n-expansion of the numbers on the interval $[0,1]$. The map $f_n(x)$ is thus a model for the behavior of such a dynamical system.

Proposition 4. *The seed value of $x_1 = 2/(n^k+1)$ induces a period k-cycle for the map $f_n(x)$.*

Consider the itinerary given by

$$< \overline{A, A, A, \ldots, A, B}, \ldots >,$$

with $k-1$ consecutive symbols A and one B such that the string repeats indefinitely. The rule for evaluating the seed number x_1 is that the expansion will repeat after $2k$ terms and

1. If an even number of B's appear before an A in the m-th position, then the expansion term in the m-th position is $0/n^m$, else it is $(n-1)/n^m$.

2. If an even number of B's appear before a B in the m-th position, then the expansion term in the m-th position is $1/n^m$, else it is $(n-2)/n^m$.

Therefore, the expansion of the first $2k$-terms is

$$\begin{aligned} 1/n^k + (n-1)/n^{k+1} + \ldots + (n-1)/n^{2k-1} + (n-2)/n^{2k} &= \\ (n^k + n^{k-1}(n-1) + n^{k-2}(n-1) \ldots + n(n-1) + n - 2)/n^{2k} &= \\ (n^k + n^k - n^{k-1} + n^{k-1} - \ldots + n^2 - n + n - 2)/n^{2k} &= \\ (2n^k - 2)/n^{2k} = 2(n^k-1)/n^{2k}. \end{aligned}$$

Taking account of the entire symbol sequence gives rise to a geometric series of the form
$$\alpha = 2(n^k - 1)\sum_{j=1}^{\infty}\Big(\frac{1}{n^{2k}}\Big)^j.$$
Applying the geometric sequence formula gives
$$\alpha = \frac{2}{n^k + 1}.$$
This completes the proof.

In particular, we observe that $x = 2/(n^3+1)$ gives a period three cycle for $f_n(x)$. Of course, for large n there are many period three cycles.

We remark that the seed value $2/(n^{k-1} + n^{k-2} \ldots + n + 1)$ also produces a period k-cycle. To obtain this, one applies the rule to the symbol sequence $<\overline{A, A, \ldots, A, B, B}, \ldots>$, where A appears $k-2$ times.

Proposition 5. *If k is prime, then the number of unique k-cycles for $f_n(x)$ is given by is $(n^k - n)/k$.*

Proof: We can see from the above exposition that any sequence of k symbols repeated indefinitely will correspond to a point which is k-periodic. In other words, $f_n^k(x_1) = x_1$ with respect to the dynamical system defined by iterates of $f_n(x)$. Suppose that there is a point which corresponds to a sequence of k symbols which is repeated indefinitely and the corresponding point x_1 is of period $r < k$, where the period r is assumed to be minimal in this sense. The symbol sequence will be of the form
$$<\overline{X_1, X_2, X_3, \ldots, X_k}, \ldots> \text{ where } X_i \in \{A_1, A_2, A_3, \ldots, A_n\},$$
and where the bar indicates repetition. The first iterate is $f(x_1) = x_2$ such that $x_2 \simeq <\overline{X_2, X_3, X_4 \ldots, X_1}, \ldots>$. Therefore
$$x_r \simeq <\overline{X_r, X_{r+1}, X_{r+2}, \ldots, X_{r+k}}, \ldots>,$$
where $X_{i+k} = X_i$ for each $i \in \mathbf{Z}^+$. If $r = 1$ then we have that $X_i = A_i$ for each $i = 1, 2, ..., k$ which is a constant for the iteration. If $1 < r < k$, then since r does not divide k we have $pr < k < (p+1)r$ for some $p \in \mathbf{Z}^+$. It follows from this that x_r is k-periodic, since it can have no shorter period.

Now observe that the total number of symbol sequences that satisfy our definition is n^k but n such sequences have period 1 and so should not be counted. Of the remaining symbol sequences each is of period k but each such sequence includes k different staring points. Therefore, the number of unique sequences is $(n^k - n)/k$.

Bibliography

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