Expection Time for Escape from a Convex Polygon

P. Coulton

Abstract

We consider the case of a convex polygon in the plane and give a general formula for the time to escape moving in a straight line when the direction for escape is given. Using this formula we can obtain the expectation for escape for the choice of a random direction by integrating over the set of possible directions. Special cases will be considered and we will show how to simulate the problem of expectation using maple.

AMS classification:

Keywords: geometric probability, expectation, polygon, set diameter

1. Segment Strategies in the Plane

We consider the following problem:

- You awake in a forest (a convex polygon K) whose geometry is completely known to you, but you remember nothing about how you came to be in the forest.
- Let σ denote a straight line strategy for escape.
- Given an escape strategy, σ, we shall call the average time for escape (where the time is identified with arc-length) the *expectation time*.

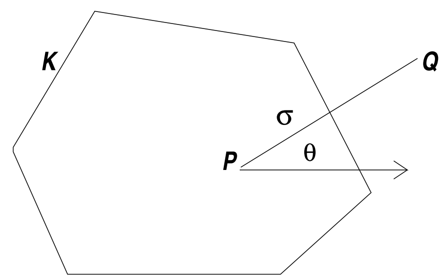

Figure 1. *An escape strategy from P in direction θ.*

Thus we have that the expectation is

$$E(\sigma) = \frac{1}{2\pi A(K)} \int_0^{2\pi} \int_K t(x, y, \theta)\ dA\ d\theta,$$

where $A(K)$ is the area of the polygon (or region) and $t(x, y)$ is the distance from (x, y) to the boundary of the region in the direction determined by θ. Let δ denote the diameter of the convex set K. We naturally have the following:

Lemma 1.1 *Assume that the diameter of K is 1 and the expectation for escape from K for some fixed strategy is*

$$E = \alpha, \text{ where } \alpha \in \mathbf{R}.$$

Then for $\hat{K} \sim K$ geometrically similar to K with diameter $\hat{\delta}$ we have $\hat{E} = \alpha\hat{\delta}$.

Now consider a fixed point (x, y) with chord length c(x,y) in the direction determined by θ. Then $c(x, y)/2$ is the average escape time for the set of points that lie along this chord. Therefore, In order to find the expected escape time for a given fixed direction determined by some θ we must integrate $c^2(h, \theta)$ for each position of height in K along the ζ-axis perpendicular to the angle θ. Let $E(\sigma, \theta)$ denote the expectation for escaping with respect to the fixed direction determined by θ, then

$$E(\sigma, \theta) = \frac{1}{2A(K)} \int_o^{h(\theta)} c^2(\zeta, \theta) d\zeta.$$

where $\zeta(\theta)$ denotes an axis in the plane that is perpendicular to the chords determined by the angle θ.

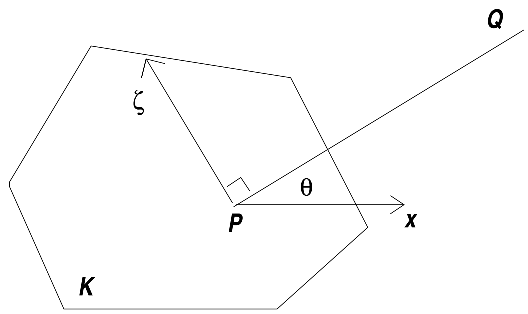

Figure 2. *Construction of the ζ-axis.*

Hence we can write

$$E(\sigma) = \frac{1}{2\pi A(K)} \int_0^{\pi} \int_0^{h(\theta)} c^2(\zeta, \theta)\ d\zeta\ d\theta,$$

where $h(\theta)$ is the altitude of the projection of K onto the ζ-axis.

Proposition 1.2 *The expectation time for a circle of diameter δ using the segment strategy is*

$$E(\sigma) = \frac{4\delta}{3\pi}.$$

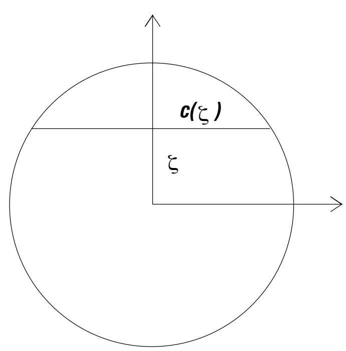

Figure 3. *Construction of the ζ-axis.*

Proof: Since the circle is symmetric with respect to the origin we have

$$E(\sigma) = \frac{4}{\pi\delta^2} \int_0^{\delta/2} \left(2\sqrt{\frac{\delta^2}{4} - \zeta^2}\ \right)^2 d\zeta = \frac{4}{\pi\delta^2} \int_0^{\delta/2} (\delta^2 - 4\zeta^2)\ d\zeta = \frac{4\delta}{3\pi}$$

$$\approx 0.424\delta.$$

2. Convex Regular Polygons

Lemma 2.1 *The expectation to escape in a given direction θ from a trapezoid with boundary lines parallel to θ is given by*

$$E(\sigma,\theta) = \frac{(c_1^2 + c_1c_2 + c_2^2)h}{6A(K)}$$

where c_1 is the length of one of the parallel sides and c_2 is the length of the opposite side and where h is the distance between the parallel sides.

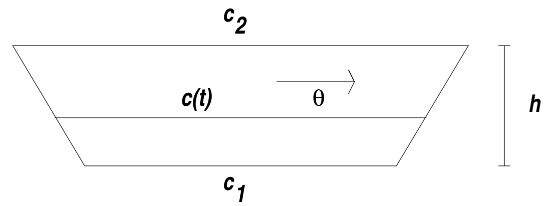

Figure 4. *An escape from the trapezoid in a direction parallel to ζ.*

Proof: Observe that the chord lengths increase in a linear fashion between points at height 0 and h. The chord length at height t is

$$c(\zeta) = c_1 + \frac{c_2 - c_1}{h}\zeta$$

The contribution of the chord integral in this region is

$$\int_o^h c^2(\zeta)d\zeta = \frac{1}{3}(c_1^2 + c_1c_2 + c_2^2)h.$$

Theorem *The expectation to escape from an n sided regular polygon with side a is given by*

$$E(\sigma) = \frac{2\tan(\pi/n)}{3a^2\pi}\int_o^{\pi/n} c_1^2h_1 + \sum_{j=2}^{n-2}(c_{j-1}^2 + c_{j-1}c_j + c_j^2)(h_j - h_{j-1})$$

$$+c_{n-2}^2(h_{n-1} - h_{n-2})\Big)d\theta,$$

where the heights are given as in Figure 5 below.

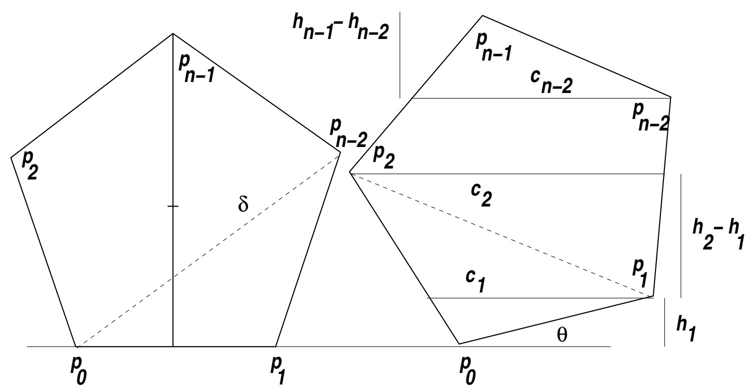

Figure 5. *Heights and chords as a function of* θ*.*

Proof: We observe that the summation over the chord values follows directly from the trapezoidal formula. We need only calculate the value of $A(K)$.

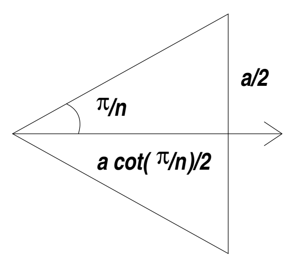

Figure 6. *The area of the triangle associated to side of length* a*.*

The area is given by

$$A(K) = a^2 n \frac{\cot(\pi/n)}{4}.$$

Plugging this into the formula and taking into account that we must complete n identical integrals from 0 to π/n gives the result.

As an example consider the case of the square of side length a. Then

$$c_o = 0,\ \ c_1 = a\sec(\theta),\ \ h_1 = a\sin(\theta), c_2 = a\sec(\theta),\ \ h_2 = a\cos(\theta) - a\sin(\theta)$$

which applied to our theorem leads to the integral formula

$$E(\sigma) = \frac{2}{3\pi a^2}\int_o^{\pi/4} a^3(3\sec(\theta) - \sec(\theta)\tan(\theta))d\theta,$$

$$= \frac{2a}{3\pi}\Big(3\ln(1+\sqrt{2}) + 1 - \sqrt{2}\Big)$$

Now consider the case of the rectangle with sides a and b. Applying the theorem for side $a_1 = a$ gives the integral formula

$$\int_o^{\arctan(b/a)} \int_o^{h(\theta)} c^2(t,\theta)\,dt\,d\theta =$$

$$\int_o^{\arctan(b/a)} \Big(-\frac{a^2 b}{3} \sec(\theta)\tan(\theta) + a^3 \sec(\theta) \Big) d\theta,$$

which leads to the formula

$$E(\sigma) = \frac{a+b}{3\pi}\Big(3\ln(1+\sqrt{2}) + 1 - \sqrt{2}\Big).$$

As a further example consider the case of the regular pentagon with side length a. We assume that for some angle θ, $0 \leq \theta \leq \pi/5$, and diameter δ we have

$$h_1 = a\sin(\theta);\ c_1 = a\cos(\theta) + a\sin(\theta)\cot(\frac{2\pi}{5} - \theta);\ h_2 - h_1 = \delta\sin(\frac{\pi}{5} - \theta);$$

$$c_2 = \delta(\cos(\frac{\pi}{5} - \theta) + \sin(\frac{\pi}{5} - \theta)\cot(\frac{\pi}{10} - \theta))$$

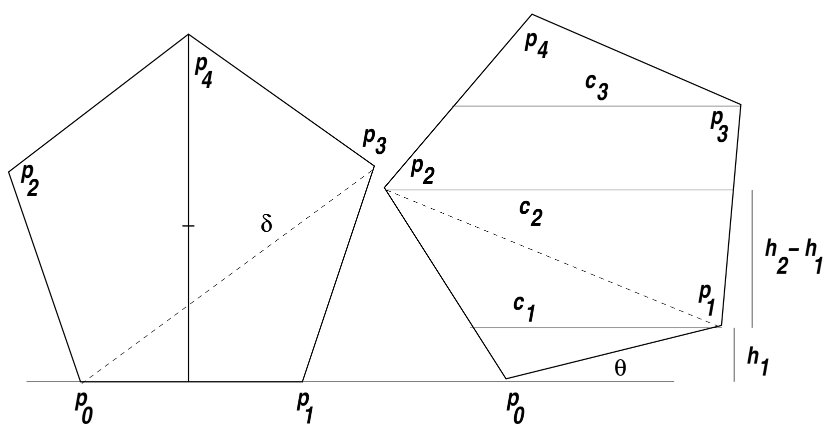

Figure 7. *An escape from the pentagon in a given direction determined by θ.*

Therefore, using symmetry and doubling the bottom half of the computation we have

$$E(\sigma) = \frac{4\tan(\pi/5)}{3a^2\pi} \int_o^{\pi/5} c_1^2 h_1 + (c_1^2 + c_1 c_2 + c_2^2)(h_2 - h_1)\,d\theta.$$

To complete the the computation, we will need to solve the following integrals:

$$\int_o^{\pi/5} c_1^2 h_1 d\theta =$$

$$a^3 \int_o^{\pi/5} \sin(\theta)\Big((\cos(\theta) + \sin(\frac{\pi}{10})\sin(\theta)\cos(\theta) + \cos(\frac{\pi}{10})\sin^2(\theta)\Big)^2 \, d\theta,$$

$$\int_o^{\pi/5} c_1^2(h_2 - h_1) d\theta =$$

$$a^2\delta \int_o^{\pi/5} \sin(\frac{\pi}{5} - \theta)\Big((\cos(\theta) + \sin(\frac{\pi}{10})\sin(\theta)\cos(\theta) + \cos(\frac{\pi}{10})\sin^2(\theta)\Big)^2 \, d\theta.$$

The last two integrals are similar and involve trigonometric functions of $\cos(\theta)$ and $\sin(\theta)$ with combined powers of 5 or less. The solution for the integrals in the case of the the general regular polygon is not significantly different since they also involve triangles and trapezoids of the same kind.

3. The Case of the General Convex Polygon.

We begin with the case of the general triangle.

Theorem *The expectation for exiting a triangle K is*

$$\begin{aligned} E(\sigma) &= \sum_{i=1}^{3} \frac{a_i \sin^2(\alpha_i)}{6\pi A(K)} \Big(\int_o^{\alpha_{i-1}} a_{i-1}^2 \sec(\frac{\pi}{2} - \alpha_{i-1} - \theta) d\theta \\ &\qquad + \int_o^{\frac{\pi}{2} - \alpha_{i+1}} a_{i+1}^2 \sec(\frac{\pi}{2} - \alpha_{i+1} - \theta) d\theta \Big), \\ &= \sum_{i=1}^{3} \frac{a_i \sin^2(\alpha_i)}{6\pi A(K)} \Big[a_{i-1}^2 \ln\Big((\sec + \tan)(\frac{\pi}{2} - \alpha_{i-1}) \Big) \\ &\qquad + a_{i+1}^2 \ln\Big((\sec + \tan)(\frac{\pi}{2} - \alpha_{i+1}) \Big) \Big], \end{aligned}$$

where the sides are of length a_i and the interior angle between a_{i-1} and a_i is α_i and $a_3 = a_o$.

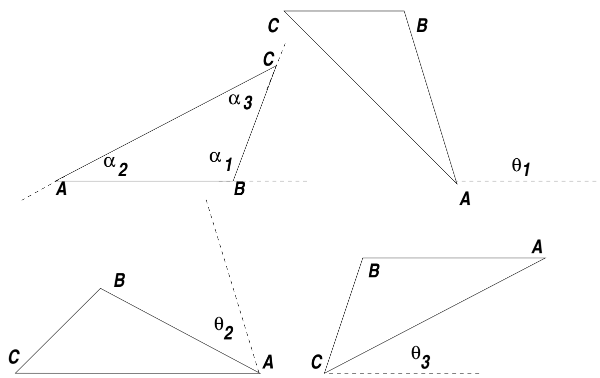

Figure 9. *An escape from a polygon in a given direction determined by θ.*

We note that the integral above is obtained by a substitution in the case of the equilateral triangle since the actual integrand is

$$\sin(\theta)\sec^2(\frac{\pi}{6}-\theta).$$

As an application to the theorem consider the case of the equilateral triangle such that that $a_1 = a_2 = a_3 = 1$ then the formula gives

$$E(\sigma) = \frac{3/4}{\pi\sqrt{3}/4}\ln(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}})$$

$$= \frac{\sqrt{3}\ln(3)}{2\pi} \approx 0.31.$$

Theorem *The expectation to escape from an n sided polygon is given by*

$$E(\sigma) = \frac{1}{6\pi A(K)}\sum_{j=1}^{n}\int_{o}^{\theta_j}\Big(c_1^2 h_1 + \sum_{i=1}^{n-3}(c_i^2 + c_i c_{i+1} + c_{i+1}^2)(h_{i+1} - h_i)$$

$$+c_{n-2}^2(h_{n-1} - h_{n-2})\Big)\, d\theta,$$

where $\sum_j \theta_j = \pi$ with the standard assumptions on the heights h_i and the chords c_i which depend on θ.

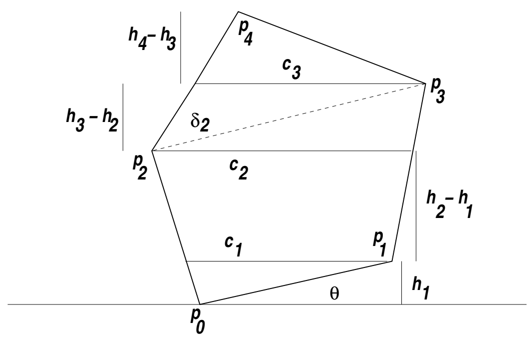

Figure 9. *An escape from a polygon in a given direction determined by θ.*

A similar formula will hold for any convex polygon whic can be decomposed as triangles and trapezoids for each angle θ. I would like to thank Steve Finch and John Wetzel for suggesting this problem to me.

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