A Theorem of Hilbert

Mat 3271 Class

Theorem (Hilbert) *Assume that there exists lines ℓ and ℓ' parallel such that ℓ is not asymptotic to ℓ'. Then there exists a unique common perpendicular to the given lines.*

Sketch of the Proof: The proof is obtained by justifying the following steps.

1. Consider points A and B on $\overleftrightarrow{AB}$ and drop perpendiculars AA' and BB' to $\overleftrightarrow{A'B'}$. If either is a common perpendicular then we are done. If $|AA'| = |BB'|$ then there exist a common perpendicular at the midpoint which is unique since there are no rectangles in $\mathcal{H}^2$.

2. On the basis of part (1), We assume WOLOG that $|AA'| > |BB'|$ and we construct E on AA' such that $EA'| = |BB'|$. (segment duplication and the embedding relation)

3. At E construct the line $\overleftrightarrow{EF}$ such that F is on SS($\overleftrightarrow{AA'}$) as B and $\angle FEA' \cong \angle GBB'$ where $A * B * G$.

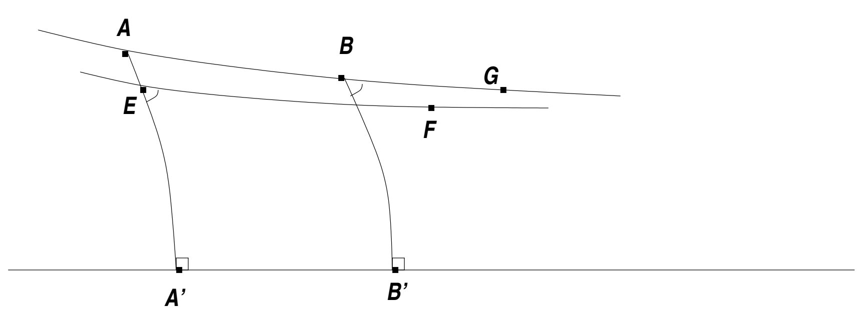

Figure 1. *Construction of $EA' \cong BB'$ and $\angle FEA' \cong \angle GBB'$.*

4. We must show that $\overleftrightarrow{EF} \cap \overleftrightarrow{AB} \neq \emptyset$. If $H = \ell(EF) \cap \overleftrightarrow{AB}$, then we can find $K \in \overleftrightarrow{AB}$, such that $BK \cong EH$.
5. If this is the case, we will show that $HH' \cong KK'$ where HH' and KK' are perp to $\overleftrightarrow{A'B'}$, which implies that the midpoint of HK on $\overleftrightarrow{AB}$ gives the point which produces the unique common perpendicular.

We say that $[ABCD$ is a biangle if $AB||CD$ and A and D are on $SS(\overleftrightarrow{BC})$. We say that BC is the base of the biangle. In addition, we will say that the biangle is closed at B if every interior ray emanating from B intersects $\overrightarrow{CD}$. Observe that AB is asymptotic to CD on the given side when the biangle is closed at B.

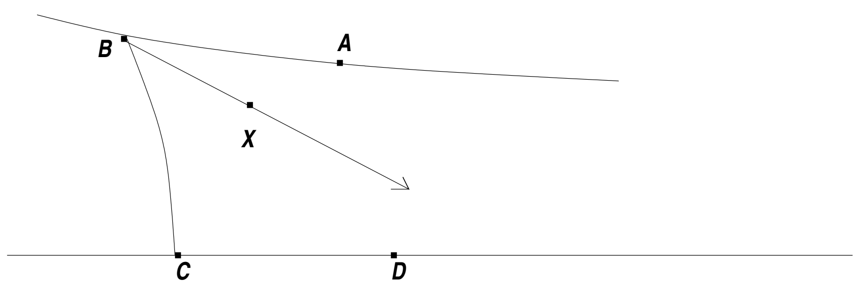

Figure 2. *An interior ray for the biangle* $[ABCD$.

To prove that such a point H exists we need the following lemmas.

Lemma 1 (Extension) *Assume that $[ABCD$ is closed at B. If $P * B * A$ or $B * P * A$ then the biangle $[APCD$ is closed at A.*

Proof:

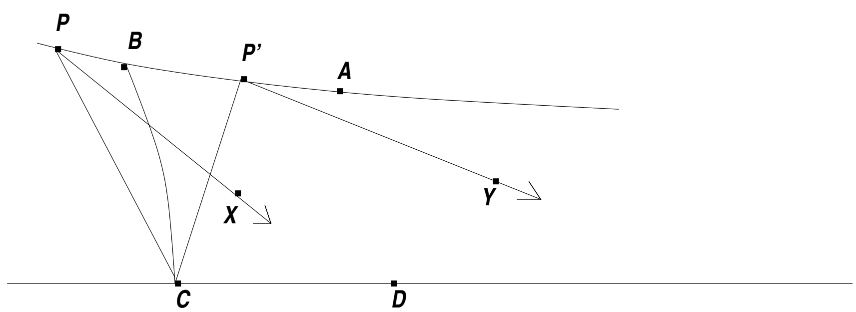

Figure 3. *Biangle closed at B implies the biangles are closed at P and P'.*

1. Extend AB to point P such that $P * B * A$
2. Show that $PX \cap CD \neq \emptyset$
3. Construct an angle at $\angle ABE$ by corresponding angles to $\angle BPX$
4. $PX \parallel CD$ (Corresponding Angles)
5. $BE \parallel PX$
6. $PX \parallel CD$ (Transitivity of Parallelism) Contradiction
7. Choose a point P' such that $B * P' * A$
8. Construct an angle at $\angle ABE$ by corresponding angles to $\angle AP'Y$
9. $BE \parallel CD$ (Corresponding Angles)
10. $P'Y \parallel BE$
11. $BE \parallel CD$ (Transitivity of Parallelism) Contradiction
12. $[ABCD$ is closed at B

Lemma 2 (Biangle Closure Symmetry) *Assume that $[ABCD$ is closed at B and show that the biangle is closed at C.*

Proof:

1. Assume by way of contradiction that $[ABCD$ is not closed at C.
2. Then some interior ray $\overrightarrow{CE}$ does not intersect $\overrightarrow{BA}$.
3. Choose $\angle BEC < \angle ECD$. This is possible by the corollary to Aristotle's axiom in Chapter 3.
4. $\overline{BE} \cap \overrightarrow{CD} = \emptyset$ because $\overrightarrow{CB} * \overrightarrow{CE} * \overrightarrow{CD}$.
5. Interior ray $\overrightarrow{BE}$ intersects $\overrightarrow{CD}$ in a point F and $B * E * F$ because $\overrightarrow{BA}|\overrightarrow{CD}$.
6. Since $\angle BEC$ is an exterior angle for $\triangle EFC$,

$$\angle BEC > \angle ECF = \angle ECD$$

   which gives us our contradiction.

7. Thus $\overrightarrow{CD}|\overrightarrow{BA}$.

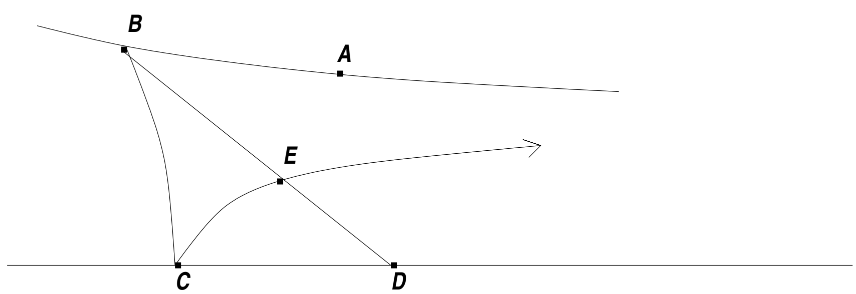

Figure 4. *To show that the biangle is closed at C.*

Lemma 3 (Inner Transitivity) *Assume that the biangles* $[BAEF$ *and* $[DCEF$ *are closed at their respective verticies and that* A *and* C *are on* $SS(\ell(EF))$*. Then the biangle* $[BACD$ *is closed.*

Proof: If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are both limiting parallel to $\overleftrightarrow{EF}$, then they are limiting parallel to each other.

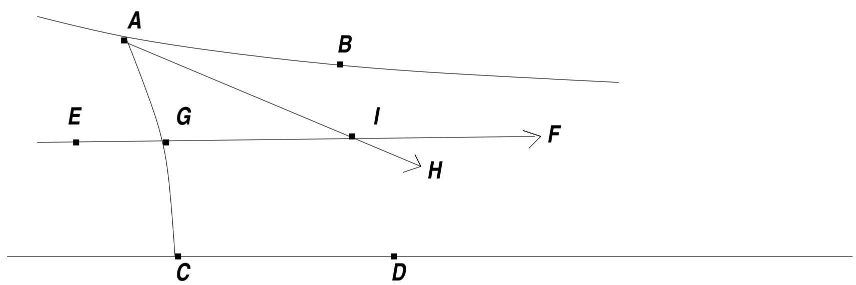

Figure 5. *Closed biangles* $[BAEF$ *and* $[DCEF$.

1. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have no point in common, by Betweenness Axiom 4: Plane Separation along line $\overleftrightarrow{EF}$.

2. Hence, there are two cases, depending on whether $\overleftrightarrow{EF}$ is between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ or $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are both on the same side of $\overleftrightarrow{EF}$, by Betweenness Axiom 3: Trichotomy.

3. In case $\overleftrightarrow{EF}$ is between $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, let G be the intersection of AC with $\overleftrightarrow{EF}$, by Betweenness Axiom 4: Plane Separation. (To switch half-planes, it would have to cross $\overleftrightarrow{EF}$.)

4. Any ray $\overrightarrow{AH}$ interior to $\angle GAB$ must intersect $\overleftrightarrow{EF}$ in a point I. Because $\overleftrightarrow{AB}$ is limiting parallel to $\overleftrightarrow{EF}$, any interior ray must intersect EF, or AB is not a limiting parallel.

5. $\overrightarrow{IH}$, lying interior to $\angle CIF$, must intersect $\overleftrightarrow{CD}$, because $\overleftrightarrow{EF}$ is limiting parallel to $\overleftrightarrow{CD}$. By symmetry of limiting parallelism, any interior ray must intersect $\overleftrightarrow{CD}$.

6. Hence, any ray $\overrightarrow{AH}$ interior to $\angle CAB$ must intersect $\overleftrightarrow{CD}$, so $\overleftrightarrow{AB}$ is limiting parallel to $\overleftrightarrow{CD}$.

Lemma 4 (Outer Transitivity) *Assume that the biangles* $[BAEF$ *and* $[DCEF$ *are closed at their respective verticies and that* A *and* C *are on* $OS(\overleftrightarrow{EF})$. *Then the biangle* $[BACD$ *is closed.*

Proof:

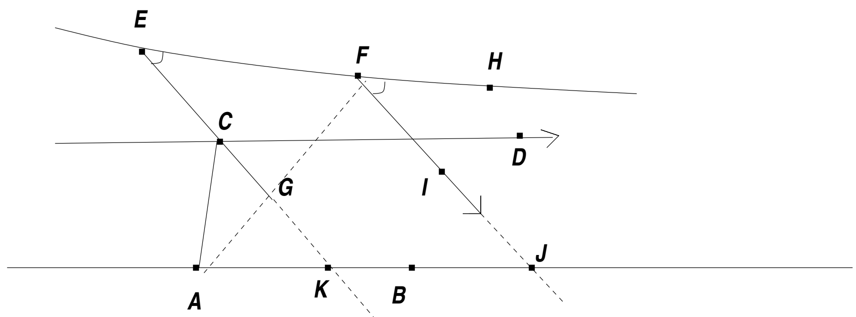

Figure 6. *Closed biangles* $[BAEF$ *and* $[DCEF$.

1. It suffices to show there is a line transversal to the three rays $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$.
2. Case 1: A and F are on the same side of $\overleftrightarrow{EC}$.
3. Then ray $\overrightarrow{EA}$ is interior to $\angle E$. Since A, C are SS($\overrightarrow{EF}$) and A, F are SS($\overrightarrow{EC}$)
4. Then $\overrightarrow{EA}$ intersects $\overrightarrow{CD}$ since $[FECD$. Thus $\overrightarrow{EA}$ is our transversal.
5. Case 2: (See figure 6) A and F are on opposite sides of $\overleftrightarrow{EC}$.
6. Let G be the point at which AF meets $\overleftrightarrow{EC}$. Since $[FEAJ$.
7. Add H such that $E * F * H$ by segment extension.
8. Thus $[HFAJ$ Since $[FEAJ$.
9. Now we have $\angle HFG > \angle E$ By exterior angle theorem.
10. Now construct ray $\overrightarrow{FI}$ interior to $\angle HFA$ such that $\angle HFI \cong \angle E$. By angle duplication.
11. Let J be the point that $\overrightarrow{FI}$ meets $\overrightarrow{AJ}$. Since $[HFAJ$.

12. Now $\overleftrightarrow{FJ} || \overleftrightarrow{EC}$ by alternate interior angle theorem.

13. Since $\overleftrightarrow{EC}$ intersects side AF and does not intersect side FJ of $\triangle AFJ$ it must intersect AJ by Pasch's Theorem.

14. Thus $\overleftrightarrow{EC}$ is our transversal.

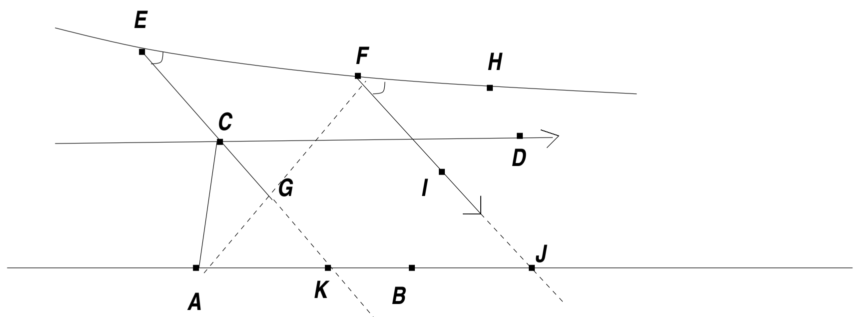

Figure 6. *Closed biangles* $[BAEF$ *and* $[DCEF$.

Lemma 5 (EA for Asymp-Δ) *Assume that $\Delta PQ\Omega$ is a singly asymptotic triangle at Ω. Then the exterior angle at P is greater than the interior angle at Q. N.B. by symmetry, the exterior angle at q is greater than the interior angle at P.*

Proof:

1. Extend PQ to a point R.
2. Given R * Q * P. We must show that $\angle RQ\Omega > \angle QP\Omega$.
3. Let $\overrightarrow{QD}$ be the unique ray on the same side of $\overleftrightarrow{PQ}$ as $\overrightarrow{Q\Omega}$ such that $\angle RQD \cong \angle QP\Omega$. (Corresponding Angles)
4. Extend $\overrightarrow{QD}$ to a point U.
5. If U * Q * P, then $\angle UQP \cong \angle QP\Omega$. (Vertical Angle Theorem)
6. By Exercise 14, $\overleftrightarrow{QD}$ and $\overleftrightarrow{P\Omega}$ are divergently parallel.
7. $\overrightarrow{QD}$ must be between $\overrightarrow{QR}$ and $\overrightarrow{Q\Omega}$.
8. If $\overrightarrow{QD}$ is between $\overrightarrow{Q\Omega}$ and $\overrightarrow{P\Omega}$, then $\overrightarrow{QD}$ meets $\overrightarrow{P\Omega}$ which is a contradiction.
9. Therefore, $\angle RQD < \angle RQ\Omega$
10. Since $\angle QP\Omega \cong \angle RQD$, Conclusion, $\angle RQ\Omega > \angle QP\Omega$.

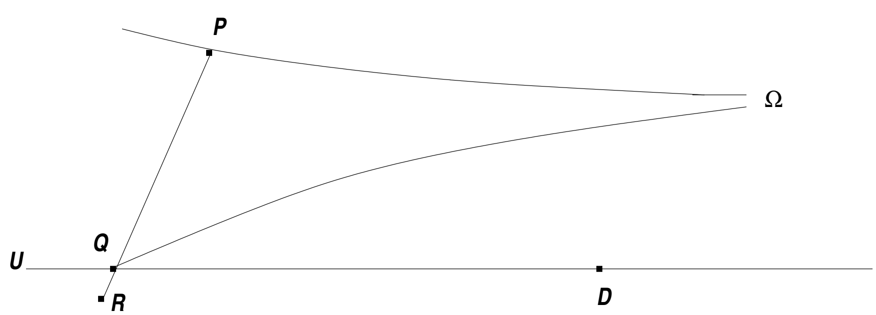

Figure 7. *The exterior angle theorem.*

Lemma 6 (Congruence of Asymp-Δ) *Assume that the singly asymptotic triangles $\Delta AB\Omega$ and $\Delta A'B'\Omega'$ satisfy $\angle BA\Omega \cong \angle B'A'\Omega'$. Then $\Delta AB\Omega \cong \delta A'B'\Omega'$ if and only if $AB \cong A'B'$.*

Proof:

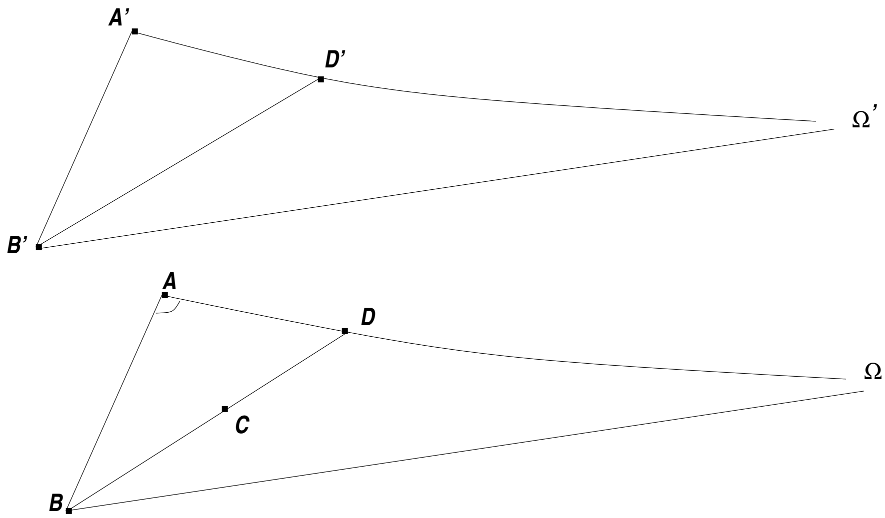

Figure 8. $\Delta ABD \cong \Delta A'B'D'$.

1. ($\Leftarrow$)Assume $|AB| \cong |A'B'|$ according to the hypothesis.
2. Also Assume $\angle AB\Omega > \angle A'B'C'$
3. There exisits a uniuque ray $\overrightarrow{BC}$ such that $\angle ABC \cong \angle A'B'\Omega'$
4. ray $\overrightarrow{BC}$ intersects $A\Omega$ at pt D
5. Let D' be the unique point on $A'\Omega'$ such that $|AD| \cong |A'D'|$
6. $\Delta BAD \cong \Delta B'A'D'$
7. $\angle ABC \cong \angle A'B'\Omega' \cong \angle A'B'D'$ which gives a contradiction.
8. ($\Rightarrow$)Assume that $\angle AB\Omega \cong \angle A'B'\Omega'$
9. Assume that $|A'B'| < |AB|$
10. Let C be the point on AB such that $BC \cong B'A'$

11. let $\overrightarrow{C\Omega}$ be the limiting ray from C to $\overrightarrow{A\Omega}$

12. $C\Omega$ is also a limiting parallel to BΩ

13. $\angle$ BC$\Omega \cong \angle$ B'C'Ω'

14. $\angle$ BA$\Omega \cong \angle$ BCΩ

15. $\angle$ BC$\Omega > \angle$ BA Ω.

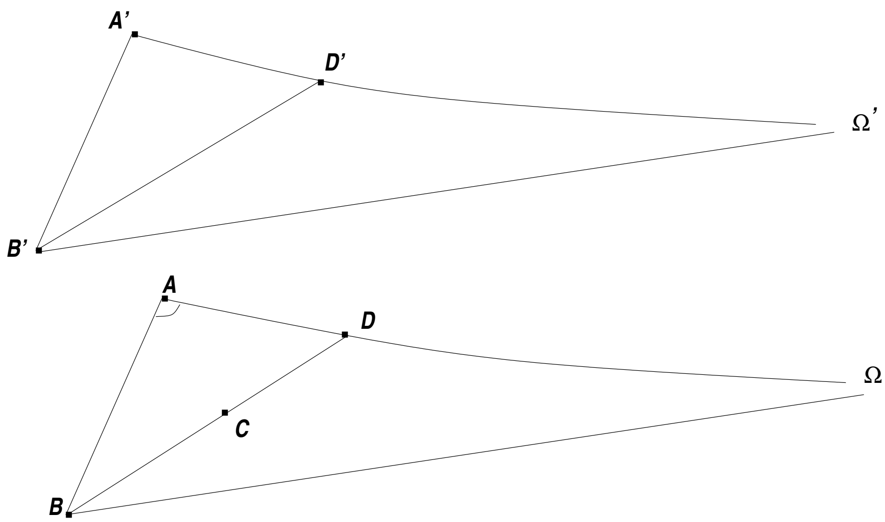

Figure 8. $\Delta ABD \cong \Delta A'B'D'$.

Lemma 7 *Assume the conditions of the theorem. Then there exists a point* $H = \ell(AB) \cap \ell(EF)$.

Proof:

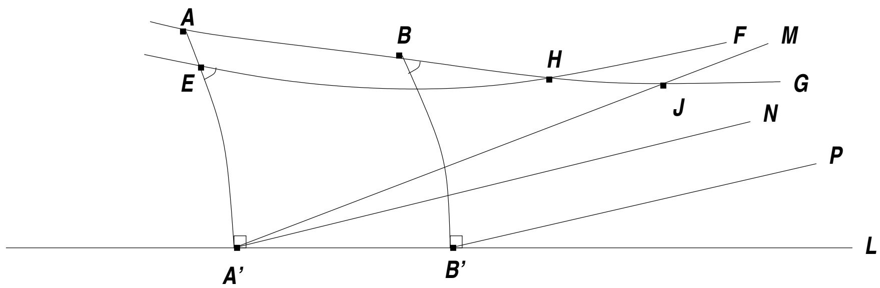

Figure 9. *Demonstration of the existence of H.*

1. Let $\overrightarrow{A'M}$ be limiting parallel to $\overrightarrow{EF}$, $\overrightarrow{A'N}$ limiting parallel to $\overrightarrow{AG}$, and $\overrightarrow{B'P}$ limiting parallel to $\overrightarrow{BG}$
2. Since $EA' \cong BB'$ and $\angle A'EF \cong \angle B'BG$, we have $\angle EA'M \cong \angle BB'P$ (side and angle gives us congruent triangles)
3. $\overrightarrow{B'L}$ differs from $\overrightarrow{A'N}$ (from previous results)
4. $\angle MA'L \cong \angle PB'L$ (by angle subtraction)
5. $\overrightarrow{B'P}$ is a limiting parallel to $\overrightarrow{A'N}$
6. Hence $\angle NA'L$ is smaller than $\angle PB'L$ (since $\angle NA'L < \angle MA'L$)
7. $A'M$ lies between $\overrightarrow{A'N}$ and $\overrightarrow{A'A}$, so it must intersect $\overrightarrow{AG}$ at a point we'll call J (since closed at A')
8.) J is on the same side of $\overrightarrow{EF}$ as A', thus on the opposite side of A

9. Thus $\overrightarrow{AJ}$ intersects $\overrightarrow{EF}$ in a point H which must be on $\overrightarrow{EF}$ because H is on the same side of $\overrightarrow{A'A}$ as J

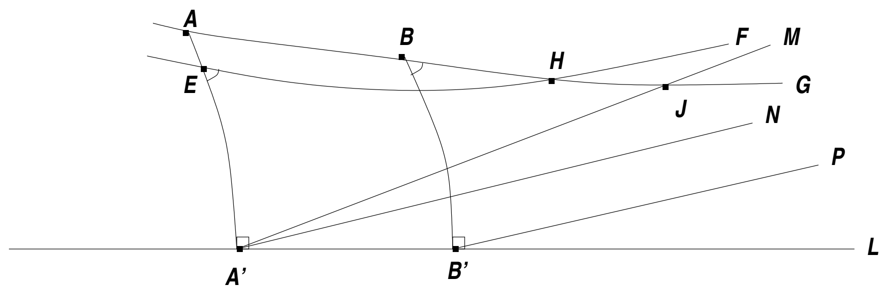

Figure 9. *Demonstration of the existence of H.*