A Theorem of Hilbert

Mat 3271 Class

Theorem (Hilbert) *Assume that there exists lines ℓ and ℓ' parallel such that ℓ is not asymptotic to ℓ'. Then there exists a unique common perpendicular to the given lines.*

Sketch of the Proof: The proof is obtained by justifying the following steps.

1. Consider points A and B on $\ell(AB)$ and drop perpendiculars AA' and BB' to $\ell(A'B')$. If either is a common perpendicular then we are done. If $|AA'| = |BB'|$ then there exist a common perpendicular at the mid-point which is unique since there are no rectangles in $\mathcal{H}^2$.

2. On the basis of part (1), We assume WOLOG that $|AA'| > |BB'|$ and we construct E on AA' such that $EA'| = |BB'|$. (segment duplication and the embedding relation)

3. At E construct the line $\ell(EF)$ such that F is on SS(AA') as B and $\angle FEA' \cong \angle GBB'$ where $A * B * G$.

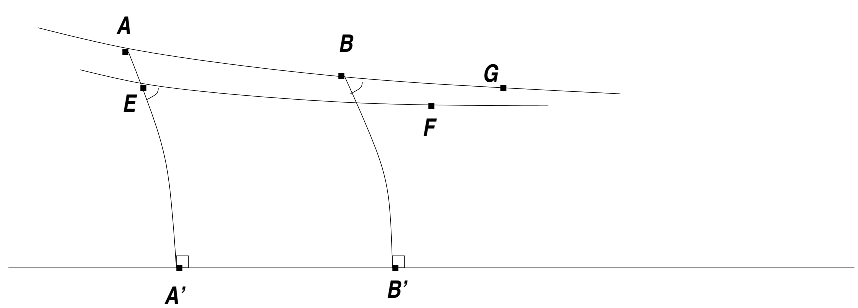

Figure 1. *Construction of $EA' \cong BB'$ and $\angle FEA' \cong \angle GBB'$.*

4. We must show that $\ell(EF) \cap \ell(AB) \neq \emptyset$. If $H = \ell(EF) \cap \ell(AB)$, then we can find $K \in \ell(AB)$, such that $BK \cong EH$.

5. If this is the case, we will show that $HH' \cong KK'$ where HH' and KK' are perp to $\ell(A'B')$, which implies that the midpoint of HK on $\ell(AB)$ gives the point which produces the unique common perpendicular.

We say that $[ABCD$ is a biangle if $AB||CD$ and A and D are on $SS(\ell(BC))$. We say that BC is the base of the biangle. In addition, we will say that the biangle is closed at B if every interior ray emanating from B intersects CD. Observe that AB is asymptotic to CD on the given side when the biangle is closed at B.

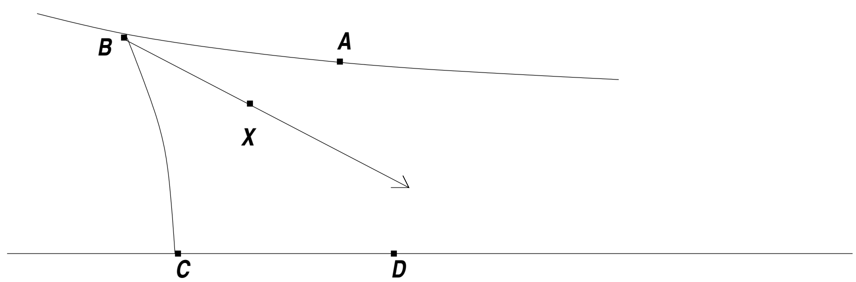

Figure 2. *An interior ray for the biangle* $[ABCD$.

To prove that such a point H exists we need the following lemmas.

Lemma 1 (Extension) *Assume that* $[ABCD$ *is closed at* B*. If* $P * B * A$ *or* $B * P * A$ *then the biangle* $[APCD$ *is closed at* A*.*

Proof: ...

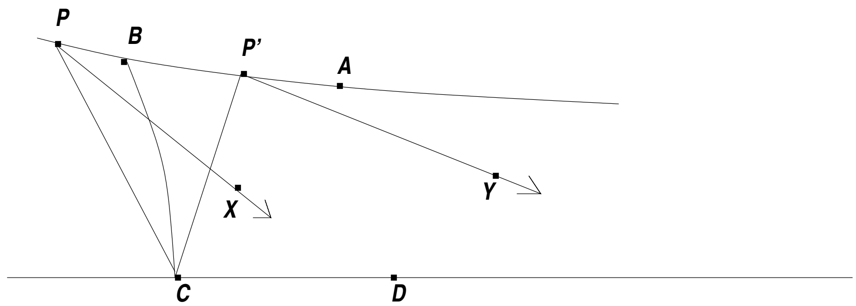

Figure 3. *Biangle closed at* B *implies the biangles are closed at* P *and* P'*.*

Lemma 2 (Symmetry) *Assume that* $[ABCD$ *is closed at* B *and show that the biangle is closed at* C*.*

Proof: ...

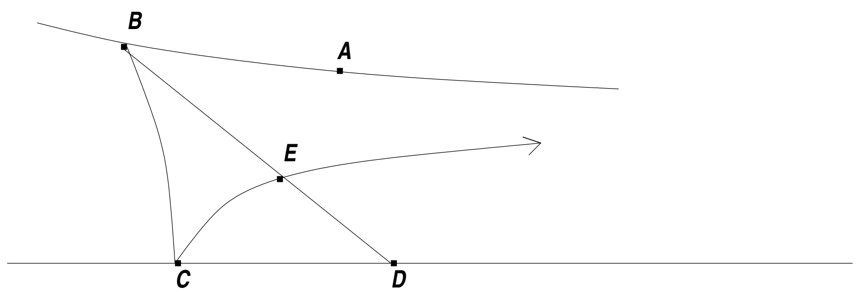

Figure 4. *To show that the biangle is closed at* C*.*

Lemma 3 (Inner Transitivity) *Assume that the biangles* $[BAEF$ *and* $[DCEF$ *are closed at their respective verticies and that* A *and* C *are on* $SS(\ell(EF))$. *Then the biangle* $[BACD$ *is closed.*

Proof: ...

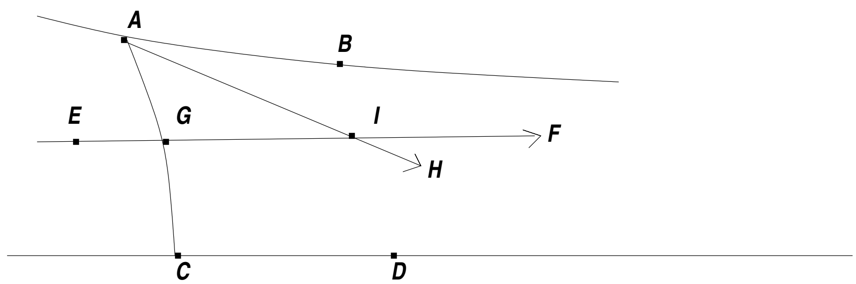

Figure 5. *Closed biangles* $[BAEF$ *and* $[DCEF$.

Lemma 4 (Outer Transitivity) *Assume that the biangles* $[BAEF$ *and* $[DCEF$ *are closed at their respective verticies and that* A *and* C *are on* $OS(\ell(EF))$*. Then the biangle* $[BACD$ *is closed.*

Proof: ...

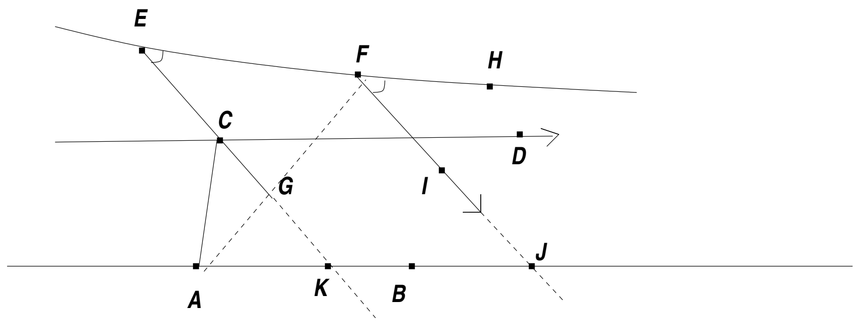

Figure 3. *Closed biangles* $[BAEF$ *and* $[DCEF$.

Lemma 5 (EA for Asymp-Δ) *Assume that $\Delta PQ\Omega$ is a singly asymptotic triangle at Ω. Then the exterior angle at P is greater than the interior angle at Q. N.B. by symmetry, the exterior angle at q is greater than the interior angle at P.*

Proof: ...

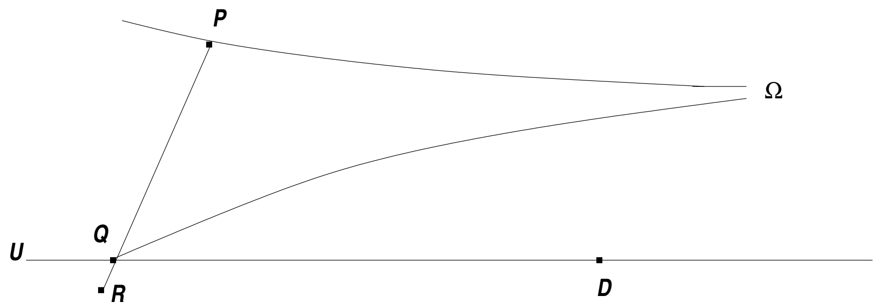

Figure 7. *The exterior angle theorem.*

Lemma 6 (Congruence of Asymp-Δ) *Assume that the singly asymptotic triangles $\Delta AB\Omega$ and $\Delta A'B'\Omega'$ satisfy $\angle BA\Omega \cong \angle B'A'\Omega'$. Then $\Delta AB\Omega \cong \delta A'B'\Omega'$ if and only if $AB \cong A'B'$.*

Proof: ...

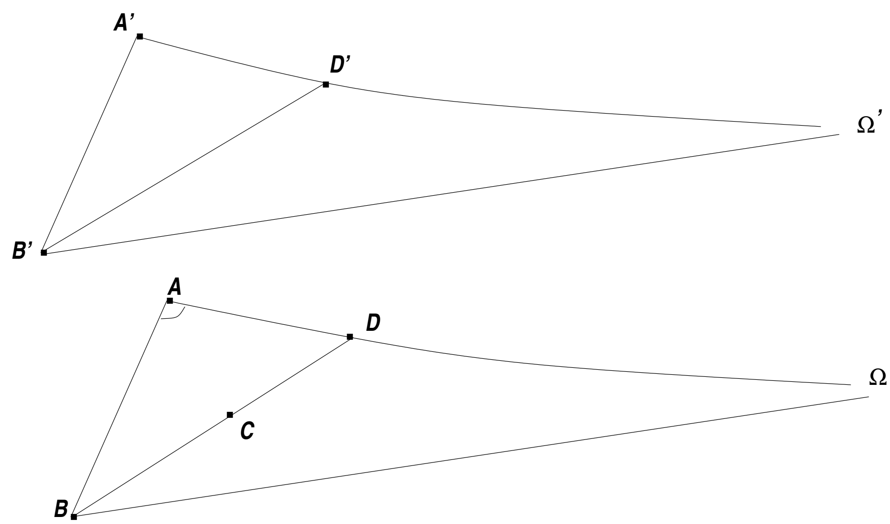

Figure 8. $\Delta ABD \cong \Delta A'B'D'$.

Lemma 7 *Assume the conditions of the theorem. Then there exists a point* $H = \ell(AB) \cap \ell(EF)$.

Proof: ...

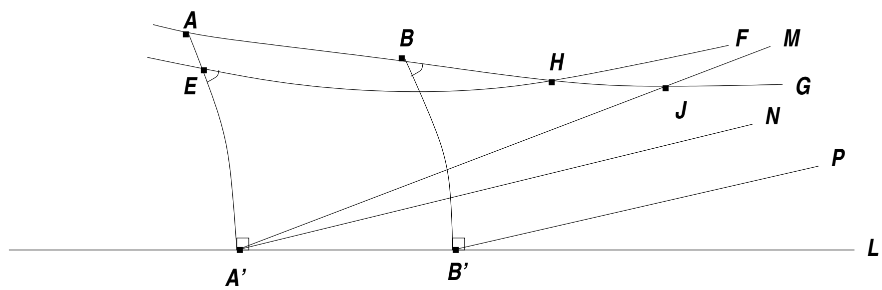

Figure 9. *Demonstration of the existence of* H.